Factoring sums of Schubert polynomials (Joint with Mahir Can, Michael Joyce)

Ben Wyser
University of Illinois at Urbana-Champaign
bwyser@illinois.edu

April 13, 2013

Basic setup

- $G = GL(n, \mathbb{C})$
- $B =$ lower-triangular Borel
- $T =$ diagonal maximal torus
- $W = N_G(T)/T = S_n$
- $H = O(n, \mathbb{C})$ or $Sp(n, \mathbb{C})$ if n is even

The weak order on B-orbit closures on G/H

Suppose that Y, Y' are two B-orbit closures on G/H. The covering relations for what is called the *weak order* on $B\backslash(G/H)$ are as follows: Y' covers Y if and only if $Y' = P_\alpha Y \neq Y$ for some simple root α.

Combinatorial models of $B\backslash(G/H)$ in our examples

When $H = O(n, \mathbb{C})$, orbits are parametrized by involutions in S_n. The lone closed orbit corresponds to w_0, and the dense orbit to the identity.

Combinatorial models of $B\backslash(G/H)$ in our examples

When $H = O(n, \mathbb{C})$, orbits are parametrized by involutions in S_n. The lone closed orbit corresponds to w_0, and the dense orbit to the identity.

When $H = Sp(2n, \mathbb{C})$, orbits are parametrized by *fixed point-free* involutions in S_{2n}. The lone closed orbit corresponds again to w_0, and the dense orbit to $(1, 2)(3, 4) \ldots (2n - 1, 2n)$.

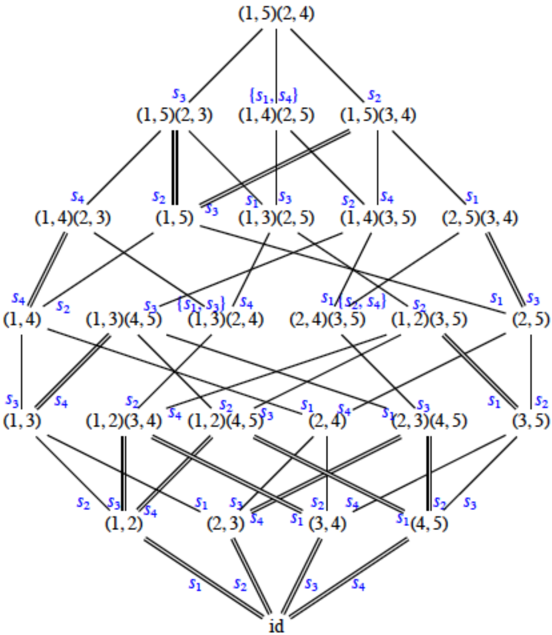
Combinatorial models of $B\backslash(G/H)$ in our examples

When $H = O(n, \mathbb{C})$, orbits are parametrized by involutions in S_n. The lone closed orbit corresponds to w_0, and the dense orbit to the identity.

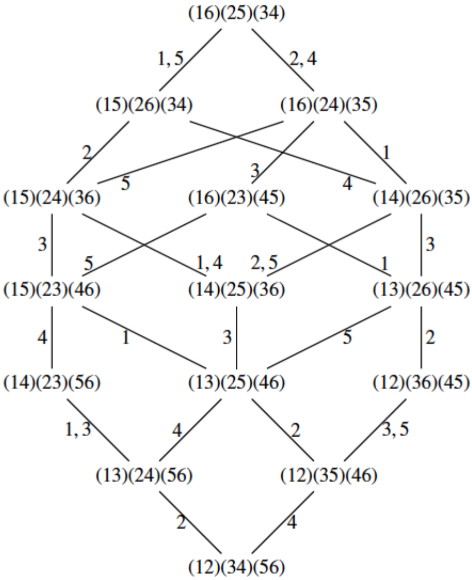
When $H = Sp(2n, \mathbb{C})$, orbits are parametrized by *fixed point-free* involutions in S_{2n}. The lone closed orbit corresponds again to w_0, and the dense orbit to $(1, 2)(3, 4) \dots (2n - 1, 2n)$.

The weak order is understood very explicitly on the level of these parametrizations.

Example 1: $(G, H) = (GL(5, \mathbb{C}), O(5, \mathbb{C}))$



Example 2: $(G, H) = (GL(6, \mathbb{C}), Sp(6, \mathbb{C}))$



The W-set of an orbit closure

Let Y be a B-orbit closure on G/H, and consider the set $W(Y) \subset W$ of distinct elements obtained by taking each path in the weak order graph from the dense orbit up to Y, and for each such path, taking the products of all edge labels. We call this the *W-set* of Y.

The W-set of an orbit closure

Let Y be a B-orbit closure on G/H, and consider the set $W(Y) \subset W$ of distinct elements obtained by taking each path in the weak order graph from the dense orbit up to Y, and for each such path, taking the products of all edge labels. We call this the *W-set* of Y.

Let $Y_{(2n+1)}$ or $Y_{(2n)}$ be the unique closed B-orbit when H is the orthogonal group. Let $Z_{(2n)}$ be the unique closed B-orbit when H is the symplectic group.

The W-set of an orbit closure

Let Y be a B-orbit closure on G/H, and consider the set $W(Y) \subset W$ of distinct elements obtained by taking each path in the weak order graph from the dense orbit up to Y, and for each such path, taking the products of all edge labels. We call this the *W-set* of Y.

Let $Y_{(2n+1)}$ or $Y_{(2n)}$ be the unique closed B-orbit when H is the orthogonal group. Let $Z_{(2n)}$ be the unique closed B-orbit when H is the symplectic group.

Q: Can we describe $W(Y_{(2n+1)})$, $W(Y_{(2n)})$, and $W(Z_{(2n)})$ explicitly?

A characterization of $W(Y_{(2n)})$ and $W(Y_{(2n+1)})$

Theorem 1 (Can-Joyce)

When $H = O(n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_n such that, in the one-line notation,

A characterization of $W(Y_{(2n)})$ and $W(Y_{(2n+1)})$

Theorem 1 (Can-Joyce)

When $H = O(n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_n such that, in the one-line notation,

- *n appears before 1, with nothing in between;*

A characterization of $W(Y_{(2n)})$ and $W(Y_{(2n+1)})$

Theorem 1 (Can-Joyce)

When $H = O(n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_n such that, in the one-line notation,

- *n appears before 1, with nothing in between;*
- *$n - 1$ appears before 2, with nothing except possibly $1, n$ in between;*

A characterization of $W(Y_{(2n)})$ and $W(Y_{(2n+1)})$

Theorem 1 (Can-Joyce)

When $H = O(n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_n such that, in the one-line notation,

- *n appears before 1, with nothing in between;*
- *$n-1$ appears before 2, with nothing except possibly $1, n$ in between;*
- *$n-2$ appears before 3, with nothing except possibly $1, 2, n-1, n$ in between;*

A characterization of $W(Y_{(2n)})$ and $W(Y_{(2n+1)})$

Theorem 1 (Can-Joyce)

When $H = O(n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_n such that, in the one-line notation,

- *n appears before 1, with nothing in between;*
- *$n-1$ appears before 2, with nothing except possibly $1, n$ in between;*
- *$n-2$ appears before 3, with nothing except possibly $1, 2, n-1, n$ in between;*
- *etc...*

Examples

- $W(Y_{(3)}) = \{312, 231\}$,

Examples

- $W(Y_{(3)}) = \{312, 231\}$,
- $W(Y_{(4)}) = \{4132, 3412, 3241\}$,

Examples

- $W(Y_{(3)}) = \{312, 231\}$,
- $W(Y_{(4)}) = \{4132, 3412, 3241\}$,
- $W(Y_{(5)}) =$
  $\{51423, 45123, 42513, 42351, 51342, 35142, 34512, 34251\}$,

Examples

- $W(Y_{(3)}) = \{312, 231\}$,
- $W(Y_{(4)}) = \{4132, 3412, 3241\}$,
- $W(Y_{(5)}) = \{51423, 45123, 42513, 42351, 51342, 35142, 34512, 34251\}$,
- etc...

A characterization of $W(Z_{(2n)})$

Theorem 2 (Can-Joyce-W)

When $H = Sp(2n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_{2n} such that, in the one-line notation,

A characterization of $W(Z_{(2n)})$

Theorem 2 (Can-Joyce-W)

When $H = Sp(2n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_{2n} such that, in the one-line notation,

- 1 *appears before* $2n$, *with nothing in between;*

A characterization of $W(Z_{(2n)})$

Theorem 2 (Can-Joyce-W)

When $H = Sp(2n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_{2n} such that, in the one-line notation,

- 1 *appears before* $2n$*, with nothing in between;*
- 2 *appears before* $2n-1$*, with nothing in between;*

A characterization of $W(Z_{(2n)})$

Theorem 2 (Can-Joyce-W)

When $H = Sp(2n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_{2n} such that, in the one-line notation,

- 1 *appears before* $2n$, *with nothing in between;*
- 2 *appears before* $2n-1$, *with nothing in between;*
- 3 *appears before* $2n-2$, *with nothing in between;*

A characterization of $W(Z_{(2n)})$

Theorem 2 (Can-Joyce-W)

When $H = Sp(2n, \mathbb{C})$, the W-set of the unique closed orbit consists of all permutations in S_{2n} such that, in the one-line notation,

- 1 *appears before* $2n$*, with nothing in between;*
- 2 *appears before* $2n - 1$*, with nothing in between;*
- 3 *appears before* $2n - 2$*, with nothing in between;*
- *etc...*

Examples

- $W(Z_{(4)}) = \{1423, 2314\}$,

Examples

- $W(Z_{(4)}) = \{1423, 2314\}$,
- $W(Z_{(6)}) = \{162534, 163425, 251634, 253416, 341625, 342516\}$,

Examples

- $W(Z_{(4)}) = \{1423, 2314\}$,
- $W(Z_{(6)}) = \{162534, 163425, 251634, 253416, 341625, 342516\}$,
- etc...

A generalization

G/H is a non-compact homogeneous space. It may be compactified in a number of ways. One particular compactification of G/H is called *wonderful*, which means that its boundary is a union of smooth G-stable divisors with smooth transveral intersections.

The wonderful compactifications of $GL(n, \mathbb{C})/O(n, \mathbb{C})$ and $GL(2n, \mathbb{C})/Sp(2n, \mathbb{C})$

When $H = O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$, the other G-orbits on the wonderful compactification of G/H are (spherical) homogeneous spaces of the form G/H_μ, where $\mu = (\mu_1, \ldots, \mu_k)$ is a composition of n, i.e. an ordered sequence of positive integers whose sum is n.

The wonderful compactifications of $GL(n, \mathbb{C})/O(n, \mathbb{C})$ and $GL(2n, \mathbb{C})/Sp(2n, \mathbb{C})$

When $H = O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$, the other G-orbits on the wonderful compactification of G/H are (spherical) homogeneous spaces of the form G/H_μ, where $\mu = (\mu_1, \ldots, \mu_k)$ is a composition of n, i.e. an ordered sequence of positive integers whose sum is n.

The group H_μ is isomorphic to

$$(O(\mu_1, \mathbb{C}) \times \ldots \times O(\mu_k, \mathbb{C})) \ltimes U$$

or

$$(Sp(2\mu_1, \mathbb{C}) \times \ldots \times Sp(2\mu_k, \mathbb{C})) \ltimes U,$$

where U is the unipotent radical of the standard parabolic subgroup corresponding to the composition μ.

Example

In the orthogonal case $GL(5,\mathbb{C})/O(5,\mathbb{C})$, for $\mu = (3,2)$, H_μ consists of matrices of the form

$$\left(\begin{array}{ccc|c} & \text{A} & & 0 \\ & & & \\ \hline * & * & * & \\ * & * & * & \text{B} \end{array}\right)$$

with $A \in O(3,\mathbb{C})$ and $B \in O(2,\mathbb{C})$.

B-orbits on G/H_μ

When $H = O(n, \mathbb{C})$, the B-orbits on G/H_μ are parametrized by *μ-involutions*, which are permutations $w \in S_n$ such that when we view the μ-blocks of the one-line notation for w as permutations (by considering only the relative order of the entries), each is an involution.

B-orbits on G/H_μ

When $H = O(n, \mathbb{C})$, the B-orbits on G/H_μ are parametrized by *μ-involutions*, which are permutations $w \in S_n$ such that when we view the μ-blocks of the one-line notation for w as permutations (by considering only the relative order of the entries), each is an involution.

For example, for $\mu = (4, 2)$, the permutations $[1642|53]$ and $[4612|35]$ are μ-involutions, since $1642 \leftrightarrow 1432$ and $53 \leftrightarrow 21$, and since $4612 \leftrightarrow 3412$ and $35 \leftrightarrow 12$.

B-orbits on G/H_μ

When $H = O(n, \mathbb{C})$, the B-orbits on G/H_μ are parametrized by *μ-involutions*, which are permutations $w \in S_n$ such that when we view the μ-blocks of the one-line notation for w as permutations (by considering only the relative order of the entries), each is an involution.

For example, for $\mu = (4, 2)$, the permutations $[1642|53]$ and $[4612|35]$ are μ-involutions, since $1642 \leftrightarrow 1432$ and $53 \leftrightarrow 21$, and since $4612 \leftrightarrow 3412$ and $35 \leftrightarrow 12$.

When $H = Sp(2n, \mathbb{C})$, B-orbits on G/H_μ are parametrized by *μ-fpf-involutions*, defined similarly.

W-sets of B-orbit closures on G/H_μ

For the case $H = O(n, \mathbb{C})$, Can-Joyce more generally give a similarly explicit description of the W-set of the closed B-orbit Y_μ for any partition μ of n, generalizing Theorem 1.

W-sets of B-orbit closures on G/H_μ

For the case $H = O(n, \mathbb{C})$, Can-Joyce more generally give a similarly explicit description of the W-set of the closed B-orbit Y_μ for any partition μ of n, generalizing Theorem 1.

Can-Joyce-W have further generalized this result to describe explicitly the W-set of *any* B-orbit closure on $GL(n, \mathbb{C})/H_\mu$ for *any* μ, further generalizing Theorem 1.

W-sets of B-orbit closures on G/H_μ

For the case $H = O(n, \mathbb{C})$, Can-Joyce more generally give a similarly explicit description of the W-set of the closed B-orbit Y_μ for any partition μ of n, generalizing Theorem 1.

Can-Joyce-W have further generalized this result to describe explicitly the W-set of *any* B-orbit closure on $GL(n, \mathbb{C})/H_\mu$ for *any* μ, further generalizing Theorem 1.

Additionally, for the case $H = Sp(2n, \mathbb{C})$, Can-Joyce-W have described explicitly the W-set of *any* B-orbit closure on $GL(2n, \mathbb{C})/H_\mu$ for *any* μ, generalizing Theorem 2.

Cohomological formulas

The closed G-orbit on $X = \overline{G/H}$ is isomorphic to G/B, so the inclusion

$$i : G/B \hookrightarrow X$$

gives rise to a restriction homomorphism

$$H^*(X) \xrightarrow{i^*} H^*(G/B).$$

Cohomological formulas

The closed G-orbit on $X = \overline{G/H}$ is isomorphic to G/B, so the inclusion

$$i : G/B \hookrightarrow X$$

gives rise to a restriction homomorphism

$$H^*(X) \xrightarrow{i^*} H^*(G/B).$$

Given any B-orbit closure Y on X, one can ask for a formula for $i^*([Y])$ either as a sum of Schubert cycles, or perhaps as a polynomial in the Chern classes x_i of (duals to) tautological subquotients which are known to generate $H^*(G/B)$.

A theorem of Brion

Theorem (Brion)

In the notation of the previous slide,

$$i^*([Y]) = \sum_{w \in W(Y)} 2^{D(w)}[X^{w^{-1}}],$$

where $D(w)$ is the number of double edges in any path in the reverse weak order graph from the bottom vertex up to Y, the product of whose edge labels is w.

A theorem of Brion

Theorem (Brion)

In the notation of the previous slide,

$$i^*([Y]) = \sum_{w \in W(Y)} 2^{D(w)}[X^{w^{-1}}],$$

where $D(w)$ is the number of double edges in any path in the reverse weak order graph from the bottom vertex up to Y, the product of whose edge labels is w.

Note: Combining with the previous theorems, this gives $i^*([Y])$ very explicitly as a sum of Schubert cycles for any B-orbit closure Y on $\overline{G/H}$. One can also give polynomials in the x_i's by summing the relevant Schubert polynomials $\mathfrak{S}_{w^{-1}}(x_1, \ldots, x_n)$ (each multiplied by the appropriate power of 2).

Alternative (factored) formulas

Can-Joyce conjectured that in the case of $H = O(n, \mathbb{C})$ and the closed B-orbit on G/H, the sum of Schubert polynomials factors nicely as

$$\sum_{w \in W(Y_{(2n)})} \mathfrak{S}_{w^{-1}}(x_1, \ldots, x_{2n}) =$$

$$x_1 \ldots x_n \prod_{1 \leq i < j \leq n} (x_i + x_j)(x_i + x_{2n+1-j}),$$

or

Alternative (factored) formulas

Can-Joyce conjectured that in the case of $H = O(n, \mathbb{C})$ and the closed B-orbit on G/H, the sum of Schubert polynomials factors nicely as

$$\sum_{w \in W(Y_{(2n)})} \mathfrak{S}_{w^{-1}}(x_1, \ldots, x_{2n}) =$$

$$x_1 \ldots x_n \prod_{1 \le i < j \le n} (x_i + x_j)(x_i + x_{2n+1-j}),$$

or

$$\sum_{w \in W(Y_{(2n+1)})} \mathfrak{S}_{w^{-1}}(x_1, \ldots, x_{2n+1}) =$$

$$x_1 \ldots x_n \prod_{1 \le i < j \le n} (x_i + x_j)(x_i + x_{2n+2-j}) \prod_{1 \le i \le n} (x_i + x_{n+1}).$$

A proof of the conjecture

B-orbits on G/H are in bijection with H-orbits on G/B, via

$$B \cdot gH \leftrightarrow H \cdot g^{-1}B.$$

A proof of the conjecture

B-orbits on G/H are in bijection with H-orbits on G/B, via

$$B \cdot gH \leftrightarrow H \cdot g^{-1}B.$$

Let $\pi_\alpha : G/B \to G/P_\alpha$ be the natural projection. The covering relations for the weak order on $H\backslash(G/B)$ are

$$Y' = s_\alpha \cdot Y$$

if $Y' = \pi_\alpha^{-1}(\pi_\alpha(Y))$.

Computing the S-equivariant class of the closed orbit

Theorem (W)

Let S be a maximal torus of H, for $H = O(2n, \mathbb{C})$, $O(2n+1, \mathbb{C})$, or $Sp(2n, \mathbb{C})$. The S-equivariant class of the closed H-orbit on G/B is represented by

$$[Y_{(2n)}] = 2^n x_1 \dots x_n \prod_{1 \leq i < j \leq n} (x_i + x_j)(x_i + x_{2n+1-j}),$$

Computing the S-equivariant class of the closed orbit

Theorem (W)

Let S be a maximal torus of H, for $H = O(2n, \mathbb{C})$, $O(2n+1, \mathbb{C})$, or $Sp(2n, \mathbb{C})$. The S-equivariant class of the closed H-orbit on G/B is represented by

$$[Y_{(2n)}] = 2^n x_1 \dots x_n \prod_{1 \le i < j \le n} (x_i + x_j)(x_i + x_{2n+1-j}),$$

$$[Y_{(2n+1)}] = 2^n x_1 \dots x_n \prod_{1 \le i < j \le n} (x_i + x_j)(x_i + x_{2n+2-j}) \prod_{1 \le i \le n} (x_i + x_{n+1}),$$

Computing the S-equivariant class of the closed orbit

Theorem (W)

Let S be a maximal torus of H, for $H = O(2n, \mathbb{C})$, $O(2n+1, \mathbb{C})$, or $Sp(2n, \mathbb{C})$. The S-equivariant class of the closed H-orbit on G/B is represented by

$$[Y_{(2n)}] = 2^n x_1 \dots x_n \prod_{1 \leq i < j \leq n} (x_i + x_j)(x_i + x_{2n+1-j}),$$

$$[Y_{(2n+1)}] = 2^n x_1 \dots x_n \prod_{1 \leq i < j \leq n} (x_i + x_j)(x_i + x_{2n+2-j}) \prod_{1 \leq i \leq n} (x_i + x_{n+1}),$$

and

$$[Z_{(2n)}] = \prod_{1 \leq i < j \leq n} (x_i + x_j)(x_i + x_{2n+1-j}),$$

in each respective case.

Comments on proof

1. The closed H-orbit Y on G/B is smooth, being isomorphic to the flag variety for H.

Comments on proof

1. The closed H-orbit Y on G/B is smooth, being isomorphic to the flag variety for H.
2. This allows us to compute the restriction of $[Y]_S$ to the S-fixed locus, using the self-intersection formula. By the localization theorem for equivariant cohomology, this restriction determines $[Y]_S$ uniquely, and any polynomial in the generators x, y which localizes correctly represents $[Y]_S$.

Comments on proof

1. The closed H-orbit Y on G/B is smooth, being isomorphic to the flag variety for H.
2. This allows us to compute the restriction of $[Y]_S$ to the S-fixed locus, using the self-intersection formula. By the localization theorem for equivariant cohomology, this restriction determines $[Y]_S$ uniquely, and any polynomial in the generators x, y which localizes correctly represents $[Y]_S$.
3. The given polynomials localize correctly.

Generalizations to other G-orbits

This approach applies equally well to the closed H_μ-orbit on G/B for any partition μ, which allows us to give (nicely factoring) polynomial representatives of the torus-equivariant classes of these orbits as well. Specializing y-variables to 0, this gives additionally a formula for the ordinary class of the orbit, equal (as polynomials) to the corresponding sum of Schubert polynomials given by our generalizations of Theorems 1-2.

Example 1

For $H = O(5, \mathbb{C})$, $\mu = (2, 3)$, the Lie algebra of the torus of G is of the form

$$\mathrm{diag}(x_1, x_2, \mid x_3, x_4, x_5),$$

while the Lie algebra of the torus of H_μ is of the form

$$\mathrm{diag}(y_1, -y_1, \mid y_2, 0, -y_2).$$

Example 1

For $H = O(5, \mathbb{C})$, $\mu = (2, 3)$, the Lie algebra of the torus of G is of the form

$$\mathrm{diag}(x_1, x_2, \mid x_3, x_4, x_5),$$

while the Lie algebra of the torus of H_μ is of the form

$$\mathrm{diag}(y_1, -y_1, \mid y_2, 0, -y_2).$$

The equivariant class of the closed H_μ-orbit on G/B is

$$2(x_1 - y_2)x_1(x_1 + y_2)(x_2 - y_2)x_2(x_2 + y_2)(x_3 + x_4)x_1x_3,$$

Example 1

For $H = O(5, \mathbb{C})$, $\mu = (2, 3)$, the Lie algebra of the torus of G is of the form

$$\mathrm{diag}(x_1, x_2, \mid x_3, x_4, x_5),$$

while the Lie algebra of the torus of H_μ is of the form

$$\mathrm{diag}(y_1, -y_1, \mid y_2, 0, -y_2).$$

The equivariant class of the closed H_μ-orbit on G/B is

$$2(x_1 - y_2)x_1(x_1 + y_2)(x_2 - y_2)x_2(x_2 + y_2)(x_3 + x_4)x_1x_3,$$

or

$$2x_1^2x_2x_3(x_1 - y_2)(x_1 + y_2)(x_2 - y_2)(x_2 + y_2)(x_3 + x_4).$$

Example 1

For $H = O(5, \mathbb{C})$, $\mu = (2, 3)$, the Lie algebra of the torus of G is of the form

$$\mathrm{diag}(x_1, x_2, \mid x_3, x_4, x_5),$$

while the Lie algebra of the torus of H_μ is of the form

$$\mathrm{diag}(y_1, -y_1, \mid y_2, 0, -y_2).$$

The equivariant class of the closed H_μ-orbit on G/B is

$$2(x_1 - y_2)x_1(x_1 + y_2)(x_2 - y_2)x_2(x_2 + y_2)(x_3 + x_4)x_1x_3,$$

or

$$2x_1^2x_2x_3(x_1 - y_2)(x_1 + y_2)(x_2 - y_2)(x_2 + y_2)(x_3 + x_4).$$

Specializing the y variables to 0, this gives

$$\sum_{w \in W(Y_{(2,3)})} \mathfrak{S}_{w^{-1}}(x_1, \ldots, x_5) = x_1^4x_2^3x_3(x_3 + x_4).$$

Example 2

For $H = Sp(6, \mathbb{C})$, $\mu = (4, 2)$, we have tori

$$\mathrm{diag}(x_1, x_2, x_3, x_4, \mid x_5, x_6)$$

and

$$\mathrm{diag}(y_1, y_2, -y_2, -y_1, \mid y_3, -y_3).$$

Example 2

For $H = Sp(6, \mathbb{C})$, $\mu = (4, 2)$, we have tori

$$\mathrm{diag}(x_1, x_2, x_3, x_4, \mid x_5, x_6)$$

and

$$\mathrm{diag}(y_1, y_2, -y_2, -y_1, \mid y_3, -y_3).$$

We get

$$\left(\prod_{i=1}^{4}(x_i + y_3)(x_i - y_3)\right)(x_1 + x_2)(x_1 + x_3).$$

Example 2

For $H = Sp(6, \mathbb{C})$, $\mu = (4, 2)$, we have tori

$$\mathrm{diag}(x_1, x_2, x_3, x_4, \mid x_5, x_6)$$

and

$$\mathrm{diag}(y_1, y_2, -y_2, -y_1, \mid y_3, -y_3).$$

We get

$$\left(\prod_{i=1}^{4}(x_i + y_3)(x_i - y_3)\right)(x_1 + x_2)(x_1 + x_3).$$

Setting y variables to 0, this gives

$$\sum_{w \in W(Z_{(4,2)})} \mathfrak{S}_{w^{-1}}(x_1, \ldots, x_6) = x_1^2 x_2^2 x_3^2 x_4^2 (x_1 + x_2)(x_1 + x_3).$$