Nonnegative sections and sums of squares

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Fundamental Problem

A polynomial $f \in S := \mathbb{R}[x_0, \ldots, x_n]$ is

- ***nonnegative*** if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$,
- a ***sum of squares*** if $f = g_1^2 + \cdots + g_k^2$ for some $g_1, \ldots, g_k \in S$.

MOTZKIN(1965): The nonnegative polynomial $x_0^4 x_1^2 + x_0^2 x_1^4 + x_2^6 - 3x_0^2 x_1^2 x_2^2$ is not a sum of squares.

PROBLEM: When is nonnegativity the same as being a sum of squares?

The Solution?

HILBERT(1888): Let S be the coordinate ring of $\mathbb{P}^n$; S has the $\mathbb{N}$-grading induced by $\deg(x_i) = 1$. If either

- $n = 1$ (univariate nonhomogeneous),
- $2d = 2$ (quadratic forms), or
- $n = 2$, $2d = 4$ (ternary quartics),

then each nonnegative $f \in S_{2d}$ is a sum of squares; else there are nonnegative $f \in S_{2d}$ that is not sums of squares.

Convex Algebraic Geometry

Fix a nondegenerate $X \subseteq \mathbb{P}^n$ such that $X(\mathbb{R})$ is Zariski dense. Let $\mathscr{O}_X(D)$ be the associated very ample line bundle.

A section $s \in H^0(X, \mathscr{O}_X(2D))$ is

- ***nonnegative*** if its evaluation at each point in $X(\mathbb{R})$ is nonnegative,
- a ***sum of squares*** if $s = \mu(t_1^2) + \cdots + \mu(t_k^2)$ for some $t_1, \ldots, t_k \in V := H^0(X, \mathscr{O}_X(D))$ where $\mu : \mathrm{Sym}^2(V) \to H^0(X, \mathscr{O}_X(2D))$.

Solution!

LEMMA: The collection of nonnegative sections (resp. sums of squares) form a closed convex cone $P_{X,2D}$ (resp. $\Sigma_{X,2D}$).

REMARK: $\Sigma^*_{X,2D}$ is a spectrahedron.

THEOREM(Blekherman-Smith-Velasco): We have $P_{X,2D} = \Sigma_{X,2D}$ if and only if $\deg(X) = 1 + \mathrm{codim}(X)$, i.e. (X, D) is a variety of minimal degree.

Toric Examples

DEL PEZZO-BERTINI(1907): A variety of minimal degree is a cone over a smooth such variety. A smooth variety of minimal degree is either

- a quadric hypersurface
- rational normal scroll, or
- the Veronese surface $\mathbb{P}^2 \subseteq \mathbb{P}^5$.

The associated Cox rings yield many new examples in which $P_{X,2D} = \Sigma_{X,2D}$.