Free Resolutions of Ideals with Binomial and Monomial Generators

Trevor McGuire

Department of Mathematics
Louisiana State University
Baton Rouge, LA

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Resolutions

If S is a ring and I is an ideal, we say $F_\bullet$ is a resolution of S/I if

$$\cdots \xrightarrow{\phi_1} F_1 \xrightarrow{\phi_0} F_0 \longrightarrow S/I$$

is an exact sequence. That is, if $\phi_{i+1} \circ \phi_i = 0$, or $\ker(\phi_i) = \operatorname{Im}(\phi_{i+1})$. The resolution is free if the F_is are all free. There are algorithms that compute free resolutions, but for certain types of ideals, we have an illuminating combinatorial algorithm.

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Buchberger Graph

Hilbert tells us that every ideal in a polynomial ring is finitely generated. In particular, for any monomial ideal, we can choose a minimal generating set, and we can use this fact to create a graph.

Definition

Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a monomial ideal with $I =< m_1, \ldots, m_r >$. The Buchberger Graph of I is a graph with r vertices labeled by the monomials and edge set $E = \{(m_i, m_j) | \mathrm{LCM}(m_i, m_j)/x_1 \cdots x_n \text{ is indivisible by } m_s \forall s\}$

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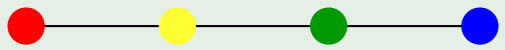
Buchberger Graph

Example

Let $I =< yz^4, xy^2z^3, x^2y^3z^2, x^3y^4z >$. Then (m_1, m_4) is not an edge because LCM$(m_1, m_4)/xyz = x^3y^4z^4/xyz = x^2y^3z^3$ is divisible by $m_2 = xy^2z^3$. But $(m_1, m_2)$is an edge as can be easily verified. The Buchberger graph for I is



where $yz^4, xy^2z^3, x^2y^3z^2, x^3y^4z$ are red, yellow, green and blue, respectively.

From the Buchberger graph, we can immediately generate a free resolution for S/I of the form

$$S^3 \xrightarrow{\phi_2} S^4 \xrightarrow{\phi_1} S \xrightarrow{\phi_0} S/I$$

The powers of the instances of S correspond to the 1 connected component, 4 vertices and 3 edges of the graph. These numbers are called the Betti numbers of I, and are invariant. We have the resolution when we know all the maps. If S^4 is generated by $s_{0,1}, s_{0,2}, s_{0,3}$ and $s_{0,4}$ as an S-module, then $\phi_1(s_{0,i}) = m_i$. If S^3 is generated by $s_{1,1}, s_{1,2}$ and $s_{1,3}$ as an S-module, then $\phi_2(s_{1,1}) = xys_{0,1} - zs_{0,2}$, $\phi_2(s_{1,2}) = xys_{0,2} - zs_{0,3}$, and $\phi_2(s_{1,2}) = xys_{0,3} - zs_{0,4}$.

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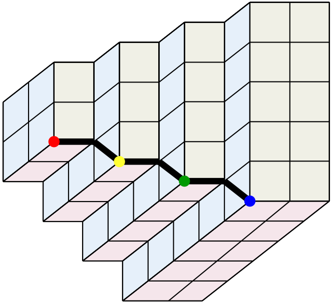

Staircase Diagrams

The fantastic property of staircase diagrams is that the Buchberger graph embeds canonically on them. That means that we need only consider staircase diagrams and not the pairwise least common multiples in the definition of the Buchberger graph.

Lattices

Lattices arise in myriad disciplines of mathematics, but in our case, we will consider lattices that are abelian subgroups of $\mathbb{Z}^n$ that intersect $\mathbb{N}^n$ only at the origin. In particular, we will consider a lattice L to be the kernel of a map $\mathbb{Z}^n \to \mathbb{Z}^m$.

Example

Let $\chi : \mathbb{Z}^3 \to \mathbb{Z}$ where $\chi(x, y, z) = 52x + 56y + 61z$. Then $L = \{(x, y, z) \in \mathbb{Z}^3 | 52x + 56y + 61z = 0\}$. In particular, $\{(4, 5, -8), (5, -9, 4), (-9, 4, 4)\} \subset L$.

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To every lattice, L, we can associate the lattice ideal $I_L =< X^{\nu^+} - X^{\nu^-} | \nu \in L >$. A minimal generating set of the lattice ideal correspond to a set called a Markov basis of the lattice. In the previous example, $I_L =< x^4y^5 - z^8, x^5z^4 - y^9, y^4z^4 - x^9 >$, so the three vectors given form a Markov basis.

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Lattice Ideal Resolutions

Just as with the monomial ideals, there is a combinatorial object called the Scarf complex that describes a minimal free resolution of lattice ideals.

The Scarf complex is a relative of the Buchberger graph, and it is natural to want to combine the two objects into an overarching generalization.

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The Scarf complex is a relative of the Buchberger graph, and it is natural to want to combine the two objects into an overarching generalization.

Buchberger Graph of Lattice Translates

We will accomplish this combination by taking the monomials in their staircase diagram and translating them around $\mathbb{Z}^n$ by the lattice generators. Then we have an infinite set of points, but we can still use the Buchberger Graph construction.

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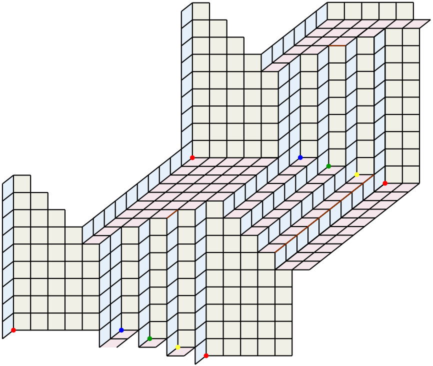
Infinite Staircase

If $M =< yz^4, xy^2z^3, x^2y^3z^2, x^3y^4z >$ and $L =< x^4y^5 - z^8, x^5z^4 - y^9, y^4z^4 - x^9 >$, then we get an infinite staircase diagram, which locally looks like



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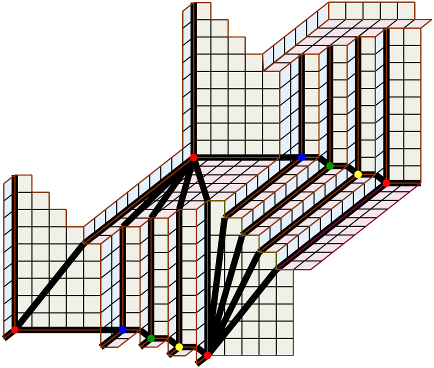
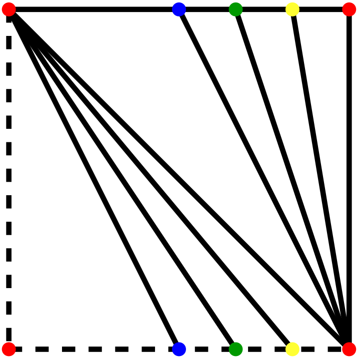
If $M =< yz^4, xy^2z^3, x^2y^3z^2, x^3y^4z >$ and $L =< x^4y^5 - z^8, x^5z^4 - y^9, y^4z^4 - x^9 >$, then the graph of $< M, I >$ has a fundamental region of



Our lattice lies inside a hyperplane in $\mathbb{Z}^3$ and under the lattice action, we can roll it to get a torus. The Buchberger Graph of Lattice Translates is exactly the graph we get under this lattice action.

The BGLT for
$I =< x^4y^5 - z^8, x^5z^4 - y^9, y^4z^4 - x^9, yz^4, xy^2z^3, x^2y^3z^2, x^3y^4z >$
is



The dotted edges indicate the repitition of edges.

This graph has 4 unique vertices, 12 edges and 8 faces, but the Betti numbers of the resolution are 1,7,14,8. This gives us a theorem.

Theorem

The Betti numbers for a minimal free resolution of $I =< M, L >$ are the sum of the Betti numbers for L with the number of unique vertices, edges and faces of BGLT(I) with a zero in the far left position.

Example

The Betti numbers of L above are 1,3,2, and the data from the BGLT(I) are 4,12, 8. So we shift the data by one to get 0,4,12,8 and add componentwise to get 1,7,14,8.

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Getting Betti numbers is good, but it isn't the full resolution. As with the Buchberger graph, though, we can read the maps directly off the graph. Since that means we can recover the monomial relations, we have another theorem.

Theorem

The Buchberger graph for M is an induced subgraph of $BGLT(< M, L >)$.

If this weren't the case, the name would be extremely misleading.This gives us a combinatorial object that reduces to the Buchberger graph if there are no lattice terms, and reduces to the Scarf complex if there are no monomials.

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The BGLT gives us a way to see both the monomial relations from the Buchberger graph as well as all the so-called mixed relations. Those are the relations that involve (preimages of) binomial terms as well as (preimages of) monomial terms. For example, one of the 14 first order relations is $yzs_{0,1} + z^5 s_{0,4} - xy^2 s_{0,7}$ where $s_{0,1}, s_{0,4}$ and $s_{0,7}$ are some generators of S^{14} and their images are $x^4y^5 - z^8, yz^4$ and x^3y^4z respectively. This relation is associated to a red-yellow edge in the BGLT.

Today we talked only about lattices in $\mathbb{Z}^3$ of dimension 2 together with monomials of $\mathbb{N}^3$. This is the highly visual case, although we can also draw pictures mutatis mutandis for lattices in $\mathbb{Z}^2$ and monomials in $\mathbb{N}^2$. The obvious generalization we would hope for would be k-dimensional lattices in $\mathbb{Z}^s$ with monomials in $\mathbb{N}^n$.

The full generalization, though, would be to construct a BGLT for any ideal that is generated by arbitrary binomial and monomial ideals. The current reliance on the Markov Basis can be limiting; it has properties we like, but the size of the Markov Basis does not behave with respect to the dimension of the lattice. Additionally, we have a genericity condition on the binomials and monomials that is increasingly difficult to satisfy in higher dimensions. These problems will be fixed with a Markovian closure and ideal perturbations, though.

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