# Matroids and stabilization of $K$-polynomials

Alex Fink
joint with Andrew Berget

North Carolina State University

Combin. and Comput. CA & AG

Boulder

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# *Dramatis personae* of varieties

Fix integers $0 \le r \le n$ throughout.

We have group actions

$$\begin{array}{l} \mathrm{GL}_r \curvearrowright \mathbb{A}^{r\times n} \curvearrowleft T = (\mathbb{C}^\times)^n \subset \mathrm{GL}_n \\ \pi\colon\ /\mathrm{GL}_r \downarrow \\ \qquad\quad \mathrm{Gr}(r,n) \curvearrowleft T \end{array}$$

The moment polytope of $\mathrm{Gr}(r,n)$ is the *hypersimplex*

$$\mathrm{conv}\left\{ \sum_{j\in B} e_j \,:\, B \subset [n], |B| = r \right\}.$$

E.g. $(r,n) = (2,4)$:




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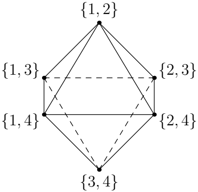


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# Matroids

Matroids extract the combinatorics of vector configurations in $(\mathbb{K}^r)^n$.
One approach: which $r$-tuples of the vectors are bases?

The moment polytope ($\Rightarrow$ iso type) of $\overline{xT} \subset \mathrm{Gr}(r,n)$ is determined by its vertices, namely which $T$-fixed points it contains (the bases).

**Definition**

A matroid is a polytope all of whose edges are edges of the hypersimplex.

**Example**




Not all matroids are realizable by vector configurations $\iff$ $T$-orbits on $\mathrm{Gr}(r,n)$.

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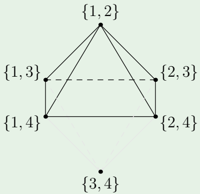


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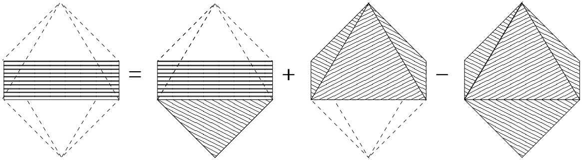
# Matroid invariants from $K$-theory

Several known matroid invariants are additive on polytope subdivisions, with inclusion-exclusion ([Tutte], [Billera-Jia-Reiner], [Derksen], [Speyer], ...)



The $K$-classes of orbits degenerate the same way.

**Theorem (Speyer)**

- *The $K$-class $\mathcal{K}(\overline{xT})$ depends only on the matroid of $x$.*
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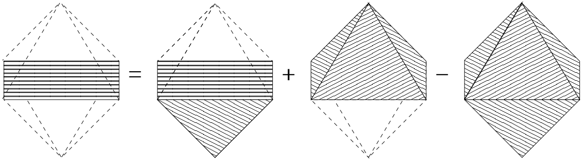

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# Orbits in $\mathbb{A}^{r \times n}$

**Question**

Does $\mathcal{K}(\overline{\mathrm{GL}\, vT})$ depend only on the matroid of $v$? (... yes)

Application.
Suppose we wish to decompose the cyclic $\mathrm{GL}_r$-module generated by a tensor $v_1 \otimes \cdots \otimes v_n \in (\mathbb{C}^r)^{\otimes n}$ into irreducibles.
(Or the Schur-Weyl dual problem with $\mathfrak{S}_n$.)

This module is dual to the $T$-degree $(1, \ldots, 1)$ piece of the coordinate ring of $\overline{\mathrm{GL}\, vT}$.

$$K^0_{\mathrm{GL} \times T}(\mathbb{A}^{r \times n}) = K^0_{\mathrm{GL} \times T}(\mathrm{pt}) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}]^{\mathfrak{S}_r}$$

while, if $L$ is the locus of lower-rank matrices,

$$K^0_T(\mathrm{Gr}(r, n)) = K^0_{\mathrm{GL} \times T}(\mathbb{A}^{r \times n} \setminus L) \twoheadleftarrow K^0_{\mathrm{GL} \times T}(\mathbb{A}^{r \times n}).$$

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# Avoiding ideals

The (equivariant $K$-theoretic) avoiding ideal $\mathcal{A}(Y)$ of an invariant subvariety $Y \subseteq X$ is the kernel of $K^0_G(X) \to K^0_G(X \setminus Y)$.

$\mathcal{A}(Y)$ is generated as a $K^0_G(\mathrm{pt})$-module by sheaves supported on $Y$.

**Lemma 1**

*Let $T \curvearrowright X$ be a torus, and $\nu : \mathrm{Char}(T) \to \mathbb{R}$. The minimum "width"*

$$\max\{\nu(\chi) : \chi \in \mathrm{supp}\, c\} - \min\{\nu(\chi) : \chi \in \mathrm{supp}\, c\}$$

*for nonzero $c \in \mathcal{A}(Y)$ is attained by $c = \mathcal{K}(Y)$.*

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# $\mathcal{K}(\overline{\mathrm{GL}\, vT})$ is a matroid invariant

[Speyer] says that $\overline{\mathrm{GL}\, vT} \in K^0_{\mathrm{GL}\times T}(\mathbb{A}^{r\times n})$ is determined by the matroid of $v$ up to an element of $\mathcal{A}(L)$.

To pin it down:

$\mathcal{K}(L)$ is a polynomial of degree $n - r + 1$ in $u_r$, so

Every nonzero element of $\mathcal{A}(L)$ which is polynomial in the $u$ variables has degree at least $n - r + 1$ in $u_r$.

**Proposition 2**

*If $Y$ is an invariant subvariety of $\mathbb{A}^{r\times n}$ not contained in $L$, then $\mathcal{K}(Y)$ is polynomial of degree at most $n - r$ in $u_r$.*

**Main theorem**

*The $K$-class $\mathcal{K}(\overline{\mathrm{GL}\, vT})$ is determined by the matroid of $v$.*

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# An orbit-raising operator

After [Fehér-Rimányi]:

Embed $\mathbb{A}^{(r-1)\times n}$ as a coordinate subspace of $\mathbb{A}^{r\times n}$.

For an invariant $Y \subset \mathbb{A}^{(r-1)\times n}$, consider $\overline{\mathrm{GL}_r\, Y} \subset \mathbb{A}^{r\times n}$.

How does the $K$-class change?

**Lemma (Fehér-Rimányi)**

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# Raising (positive) sheaves

Suppose $M$ is a $\mathbb{C}[\mathbb{A}^{(r-1)\times n}]$-module with a presentation

$$0 \to N \to \mathbb{C}[\mathbb{A}^{(r-1)\times n}] \otimes V \to M \to 0$$

for $V$ a f. d. $\mathrm{GL}_{r-1} \times T$ rep'n. Let $M'$ be given by

$$0 \to \mathrm{GL}_r\, N + (\mathrm{Minors}_r \otimes V') \to \mathbb{C}[\mathbb{A}^{r\times n}] \otimes V' \to M' \to 0.$$

where $V'$ is the rep'n with the "same" character as $V$.

The support of $M'$ is $\overline{GL_r\ \mathrm{supp}(M)}$.

**Proposition**

*The finely graded Hilbert series* $\mathrm{Hilb}(M')$ *and* $\mathrm{Hilb}(M)$ *are "the same".*

"Same" here is with respect to the Schur basis:
$s_\lambda(u_1, \ldots, u_{r-1})$ becomes $s_\lambda(u_1, \ldots, u_r)$.

# Raising on $K$-polynomials

**Theorem**

*For $M$ and $M'$ as before, we have $\mathcal{K}(M') = \rho\,\mathcal{K}(M)$, where*

$$\rho\, s_\lambda(u_1, \ldots, u_{r-1}) = \sum_{k=0}^{n} e_k(-t)\, s_{(\lambda+1,k)}(u_1, \ldots, u_r),$$

*extended $\mathbb{Z}[t^{\pm}]$-linearly.*

Interpret $s_{\lambda,k}$ by the determinantal formula.

**Proof of Proposition 2.**

Suppose $Y \subset A^{r\times n}$ not contained in $L$ had too wide a $K$-class.

Extract the coeff $c$ of the top power of $u_r$ in $\mathcal{K}(Y)$.
Then $\rho\, c$ has the same leading coefficient but is in $\mathcal{A}(Y)$.
Thus, $\mathcal{K}(Y) - \rho\, c$ contradicts Lemma 1. $\square$

# Cohomology

The raising operator translates to cohomology well, though now we pass from $r \times n$ to $(r+c) \times n$:

$$s_\lambda(u) \quad \mapsto \quad (-1)^{cr} \sum_{k_1,\ldots,k_c} e_{k_1}(t) \cdots e_{k_c}(t)\, s_{\lambda, n+1-k_1, \ldots, n+c-k_c}(u).$$

It's a corollary of the $K$-theory that

The cohomology class $\mathcal{C}(\overline{\mathrm{GL}\, vT})$ is determined by the matroid of $v$.

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**Theorem**

*$\mathcal{C}(\overline{\mathrm{GL}\, vT})$ is determined by $\mathcal{C}(\overline{\pi(v)\, T})$.*

We use this to give a formula for $\mathcal{C}(\overline{\mathrm{GL}\, vT})$ when $v$ is generic:

$$\mathcal{C}(\overline{\mathrm{GL}\, vT}) = \omega(s_{(r-1)^{n-r-1}}(u, u, t))$$

where $\omega$ transposes Schurs in $u$.

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# Other parts of the work

Interpretation of some coefficients. For $A \subseteq [n]$, the coefficient of

$$s_{(\text{hook with } a \text{ boxes and length } k)}(u) \cdot t^A$$

in $\mathcal{K}(\overline{\mathrm{GL}\, vT})$ is $(-1)^k$ if $a = |A|$ and $A$ is a rank $k-1$ dependent set, else 0.

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