Searching for exceptional mechanisms via fiber products

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Joint work with
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AMS Spring Western Sectional Meeting
Session on Comb. & Comp. Comm. Alg. & Alg. Geom. (Abo, Teitler, & Woo)

Plan for this talk

1. Finding exceptional sets via fiber products
2. Connection to kinematics
3. Numerical methods for polynomial systems
4. Progress and future plans

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<u>Basic references:</u>

A.J. Sommese & C.W. Wampler. Exceptional sets and fiber products. *Foundations of Computational Math*. 8, 171-196, 2008.

A.J. Sommese & C.W. Wampler. *The numerical solution of systems of polynomial equations arising in engineering and science.* World Scientific, 2005.

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Finding exceptional sets via fiber products

Consider the simple parameterized polynomial system

$$ax^2 + bx + c = 0$$

in parameters (a,b,c) and variable x.

Problem: Decompose the 3-dimensional parameter space according to the dimension of the solution set. In particular, find parameter values with exceptionally high-dimensional solution sets (exceptional sets of the parameter space).

Finding exceptional sets via fiber products

$$ax^2 + bx + c = 0$$

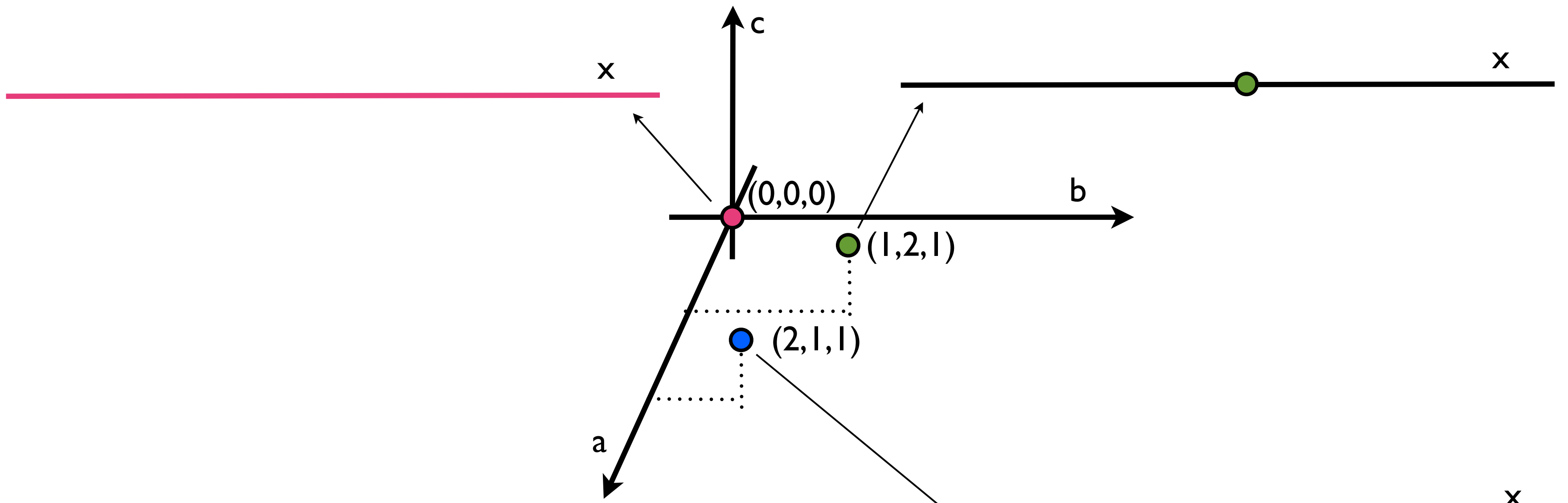
There are 5 possibilities:

1) General (a,b,c): 2 isolated solutions
2) $b^2 - 4ac = 0$: 1 isolated solution
3) $a = 0$: 1 isolated solution
4) $a=b=0$: no solutions
5) $a=b=c=0$: line of solutions (an exceptional set!!)

Easy example….

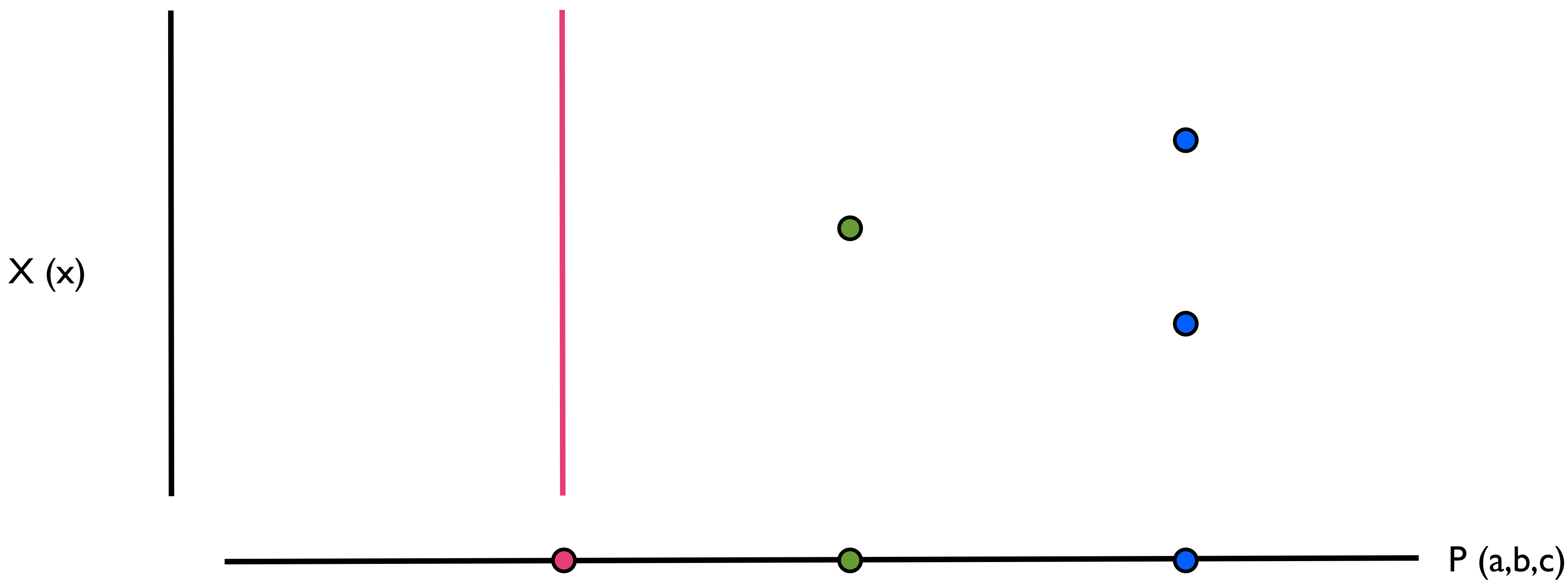
Q: How can we automate this?
A: Fiber products!

Finding exceptional sets via fiber products



Finding exceptional sets via fiber products

Schematically:



Finding exceptional sets via fiber products

If you solve

$$ax^2 + bx + c = 0$$

treating parameters (a,b,c) and variable x ALL as variables, you find a single 3-dimensional algebraic set with:

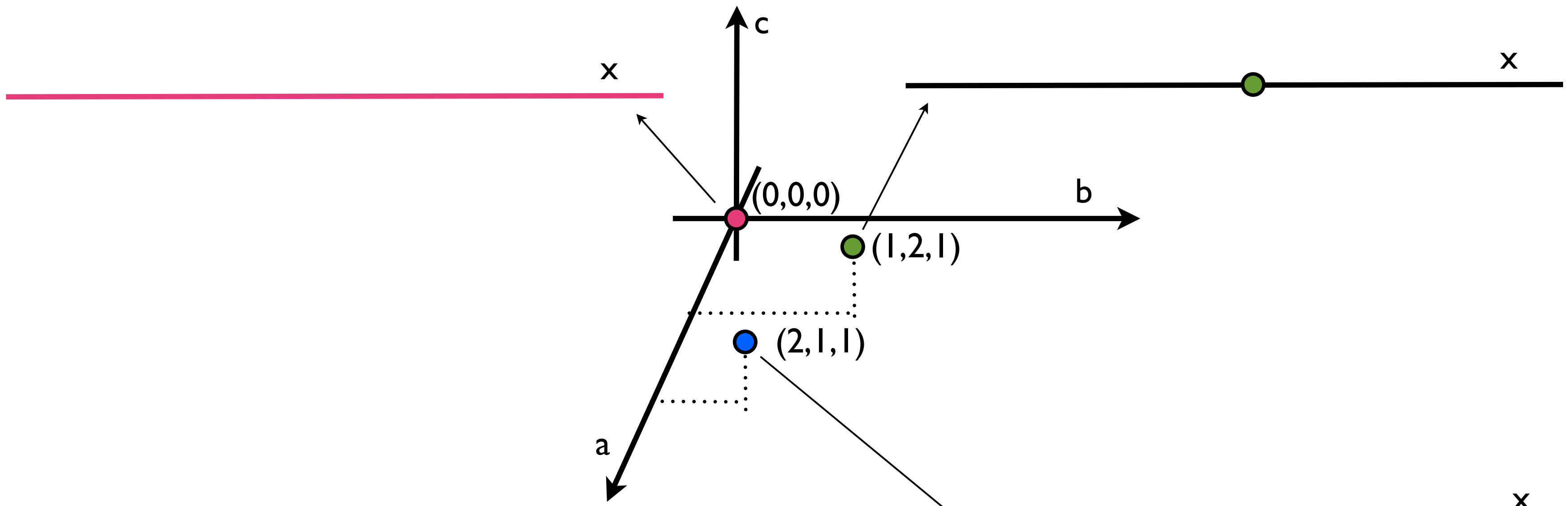
generic base dimension b = 3 and
generic fiber dimension h = 0.

KEY POINT: The exceptional fiber is buried inside! It has:

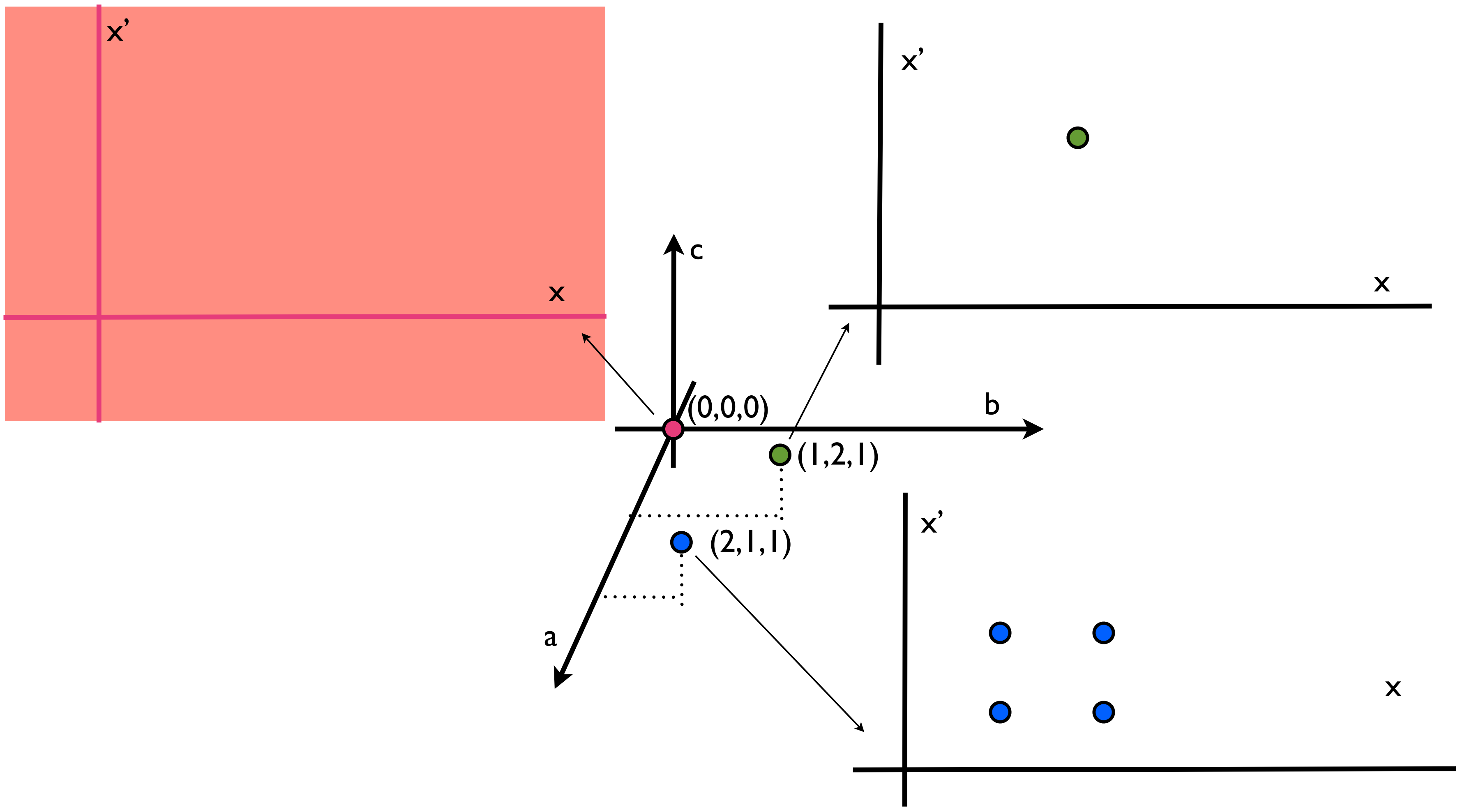
special base dimension b' = 0 and
special fiber dimension h' = 1

So, we take a *fiber product*, a product in the fiber (variable) direction:

Finding exceptional sets via fiber products

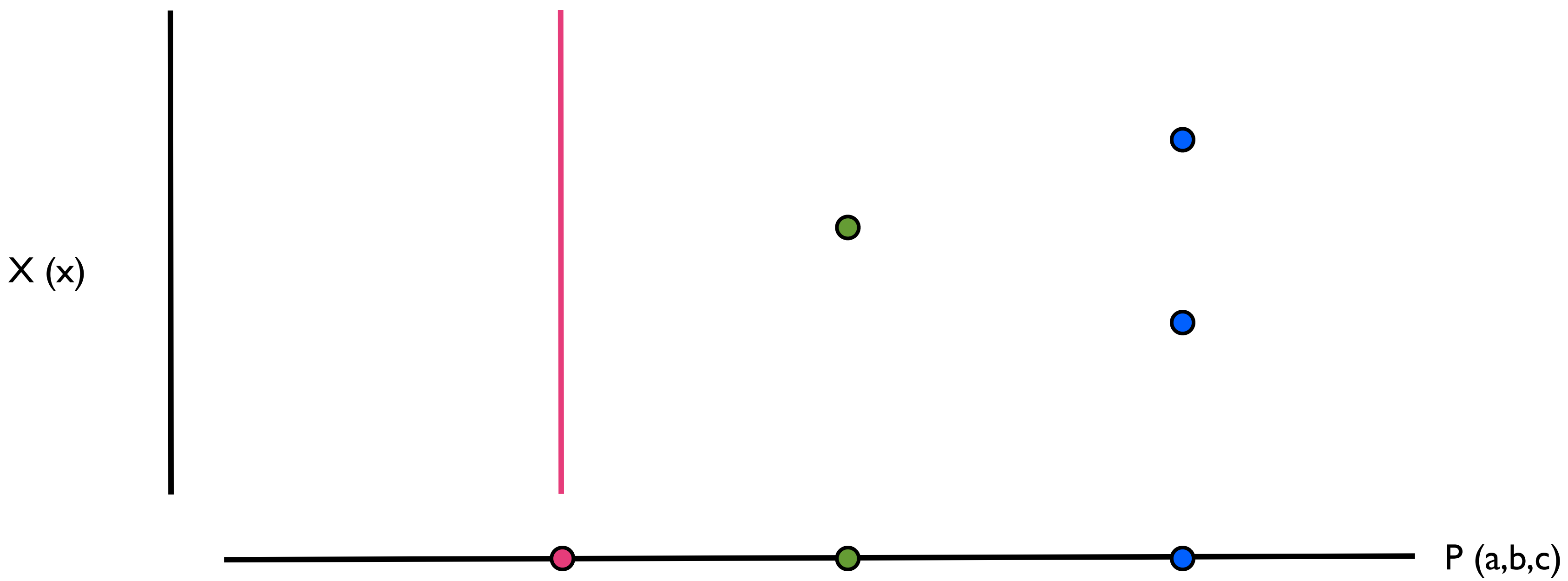


Finding exceptional sets via fiber products



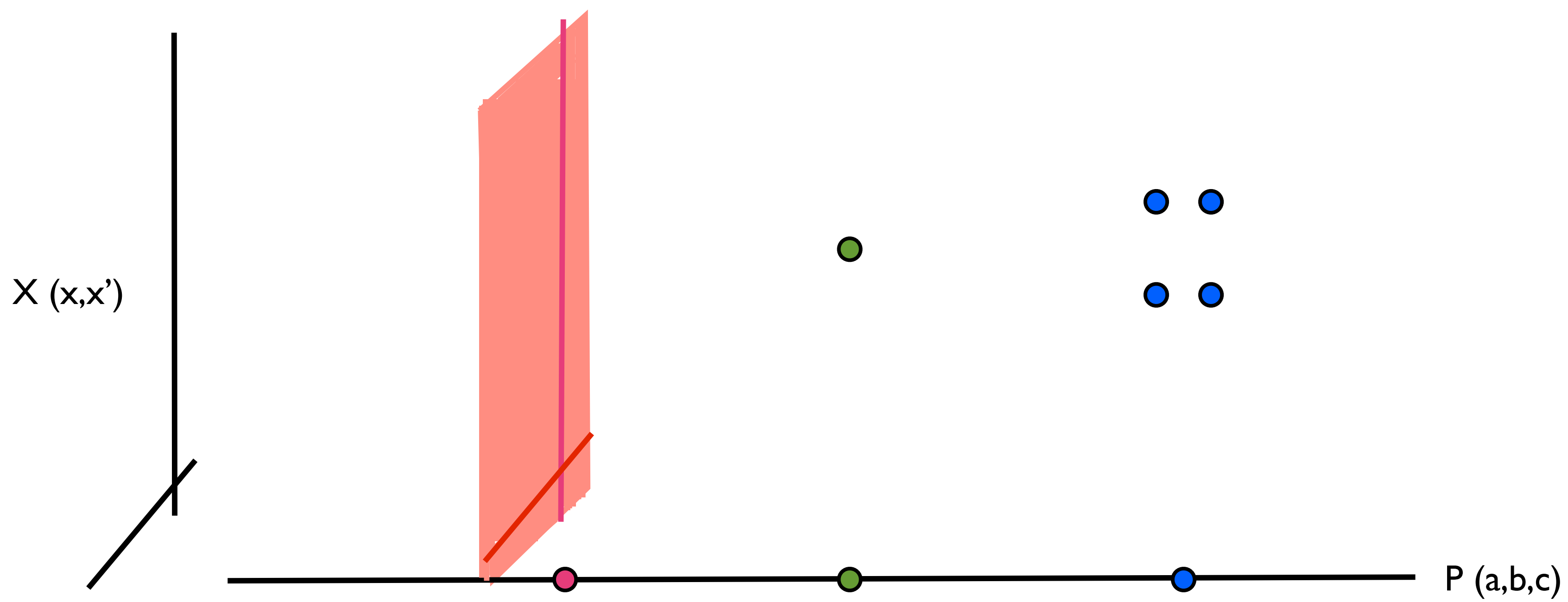
Finding exceptional sets via fiber products

Schematically (before the fiber product):



Finding exceptional sets via fiber products

Schematically (after the fiber product):



Finding exceptional sets via fiber products

At this point, we have a 2-dimensional exceptional set, still hidden inside a 3-dimensional (possibly reducible) algebraic set.

Now we have:

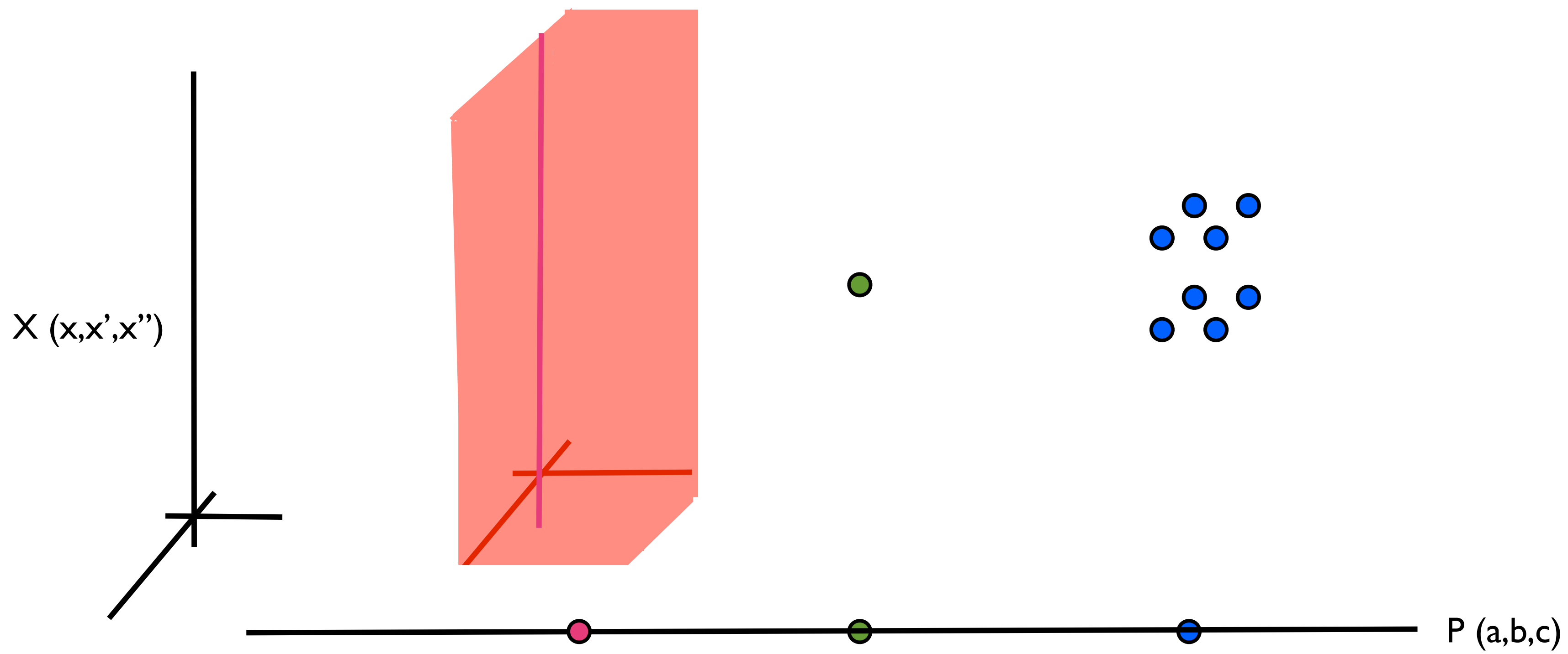
$b = 3$ $h = 0$

$b' = 0$ $h' = 2$ (h' was 1 before..it's growing....)

This is progress, but let's try another fiber product.

Finding exceptional sets via fiber products

Schematically (after the fiber product):



Finding exceptional sets via fiber products

We finally have:

$b = 3 \quad h = 0 \quad (\text{total} = 3)$

$b' = 0 \quad h' = 3 \quad (\text{total} = 3)$

so the special fiber is no longer hidden within the *generic set*.

<u>KEY POINT</u>: After k fiber products (k = number of X factors), we have:

generic set dimension: $b+kh$

special set dimension: $b'+kh'$ (where $h' > h$).

So the key is to increase k until $b'+kh' > b+kh$.

Q: So why would a kinematician care?

Q: Also, how do we do these computations??

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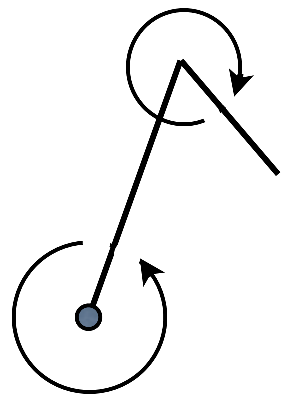
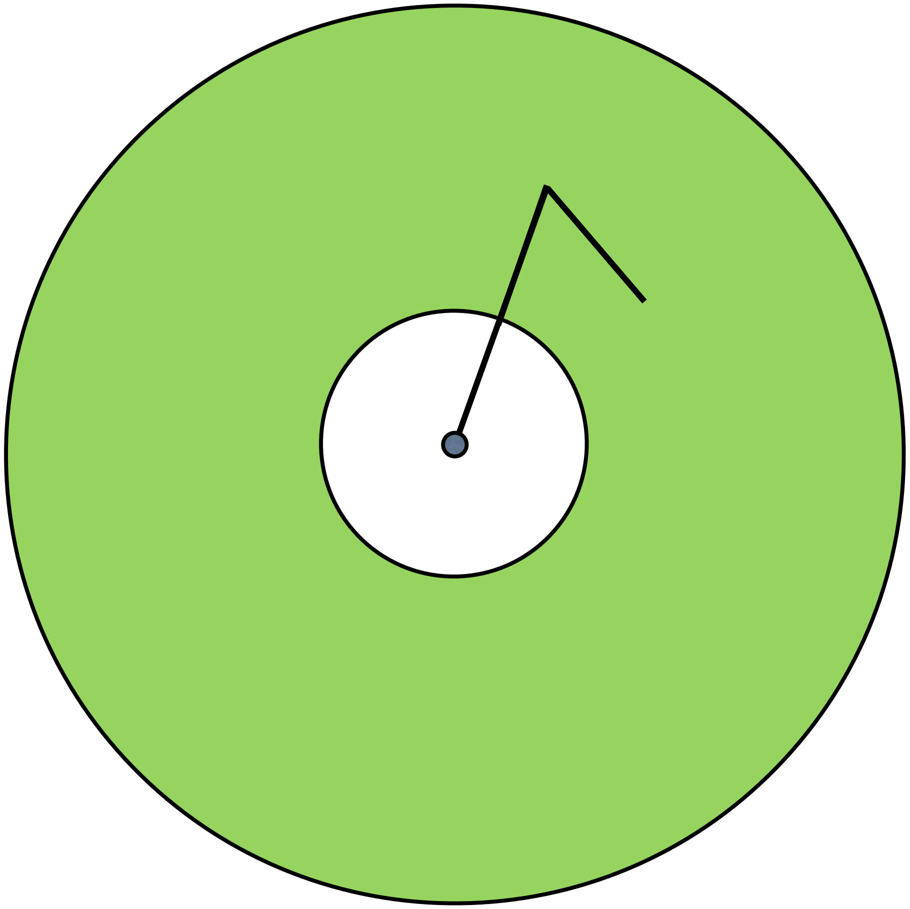
Connection to kinematics



This is a 2-link planar mechanism with rotational joints.

Here's the workspace:

Connection to kinematics



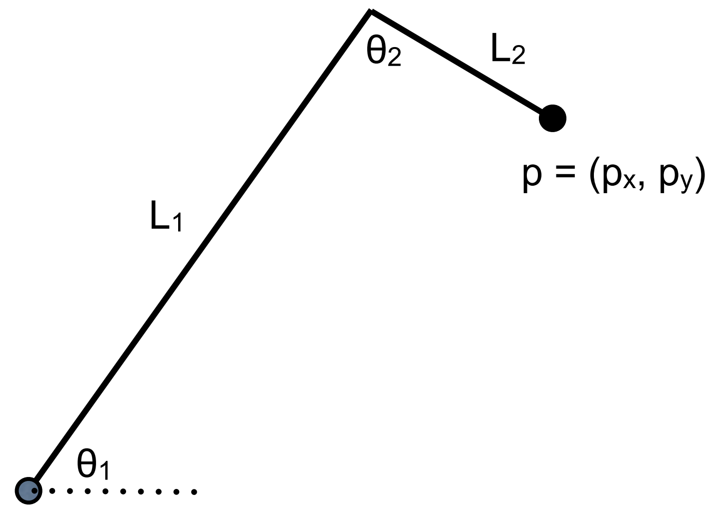
Connection to kinematics

A polynomial system describes the ability of the end effector to reach a particular point in space:

$c_1L_1 + c_2L_2 = p_x$
$s_1L_1 + s_2L_2 = p_y$
$c_1^2 + s_1^2 = 1$
$c_2^2 + s_2^2 = 1$

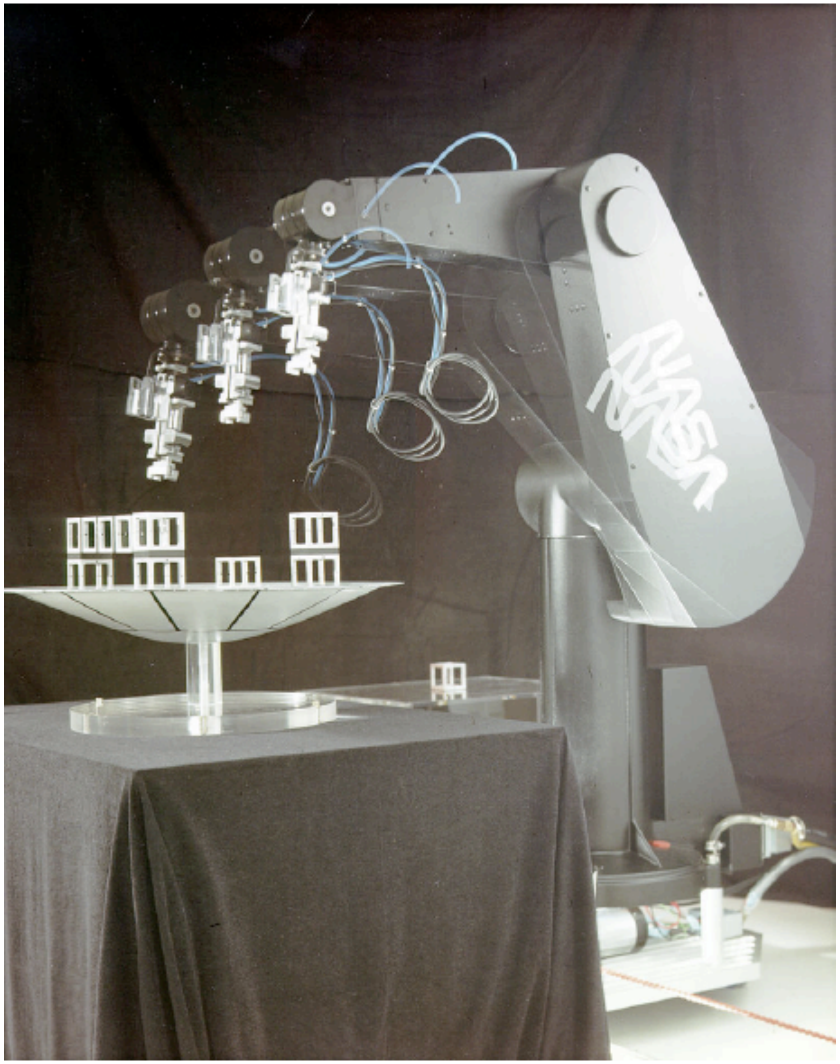
where

$c_1 = \cos(\theta_1)$ $s_1 = \sin(\theta_1)$
$c_2 = \cos(\theta_2)$ $s_2 = \sin(\theta_2)$



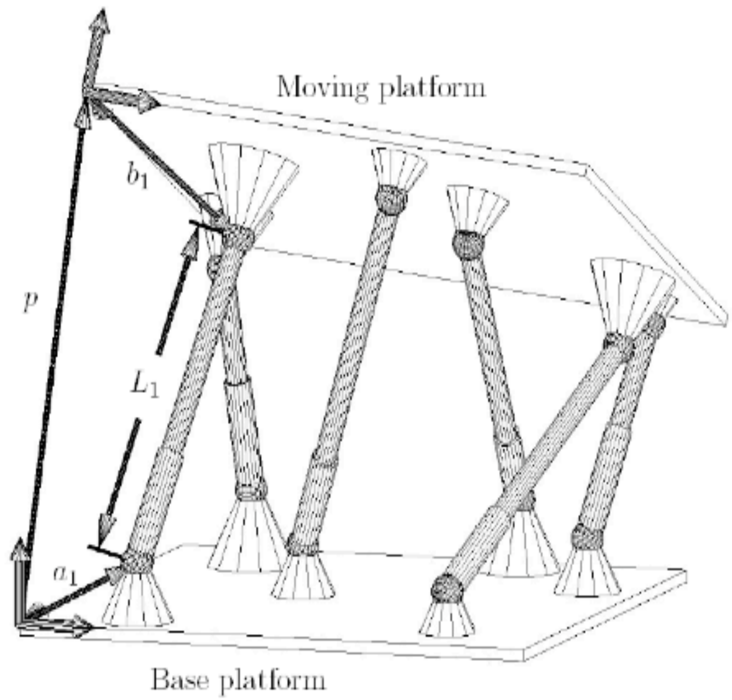
This is true of other mechanisms, too, like the PUMA:

Connection to kinematics



$$
\begin{aligned}
f_1 =&0.4318c_1c_2 - 0.2435s_1 + 0.0934as_1 + 0.4331c_1c_2s_3 + \\
&0.4331c_1c_3s_2 + 0.0203ac_1s_2s_3 - 0.0203ac_1c_2c_3 - \\
&(0.4318c_4c_5 - 0.1501s_4 - 0.0203c_4c_5c_6 + \\
&0.4331c_4c_5s_6 + 0.4331c_4c_6s_5 + 0.0203c_4s_5s_6 + \delta) \\
f_2 =&0.2435c_1 - 0.0934ac_1 + 0.4318c_2s_1 + 0.4331c_2s_1s_3 + \\
&0.4331c_3s_1s_2 + 0.0203as_1s_2s_3 - 0.0203ac_2c_3s_1 - \\
&(0.1501c_4 + 0.4318c_5s_4 - 0.0203c_5c_6s_4 + \\
&0.4331c_5s_4s_6 + 0.4331c_6s_4s_5 + 0.0203s_4s_5s_6) \\
f_3 =&0.4331c_2c_3 - 0.4318s_2 - 0.4331s_2s_3 + 0.0203ac_2s_3 + \\
&0.0203ac_3s_2 - (0.4331c_5c_6 - 0.4318s_5 + 0.0203c_5s_6 + \\
&0.0203c_6s_5 - 0.4331s_5s_6) \\
f_4 =&0.4318c_1c_2 - 0.1501s_1 - 0.0203c_1c_2c_3 + \\
&0.4331c_1c_2s_3 + 0.4331c_1c_3s_2 + 0.0203c_1s_2s_3 - x \\
f_5 =&0.1501c_1 + 0.4318c_2s_1 - 0.0203c_2c_3s_1 + \\
&0.4331c_2s_1s_3 + 0.4331c_3s_1s_2 + 0.0203s_1s_2s_3 - y \\
f_6 =&0.4331c_2c_3 - 0.4318s_2 + 0.0203c_2s_3 + \\
&0.0203c_3s_2 - 0.4331s_2s_3 - z \\
f_7 =&s_1^2 + c_1^2 - 1 \\
f_8 =&s_2^2 + c_2^2 - 1 \\
f_9 =&s_3^2 + c_3^2 - 1 \\
f_{10} =&s_4^2 + c_4^2 - 1 \\
f_{11} =&s_5^2 + c_5^2 - 1 \\
f_{12} =&s_6^2 + c_6^2 - 1
\end{aligned}
$$

Connection to kinematics



General Stewart-Gough platform robot: from Sommese & Wampler: *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*, World Scientific (2005).

This is true of Stewart-Gough platforms, too.

These are used (with actuators on the legs) for flight simulators and telescopes.

With fixed leg lengths, these are almost always rigid, but there are exceptional parameter values (Griffis-Duffy platforms) that permit motion.

Connection to kinematics

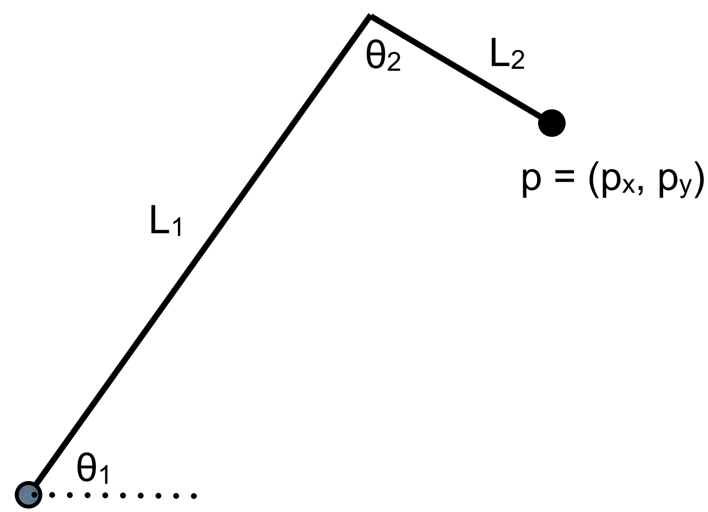
Back to the planar 2-link:

We have 4 parameters (L_1, L_2, p_x, p_y) and 4 variables (c_1, c_2, s_1, s_2).

For a general point, we expect 2 isolated solutions. For two circles of points, there are isolated solutions.

With 4 parameters, we expect a 4-dimensional irreducible component (the generic set).

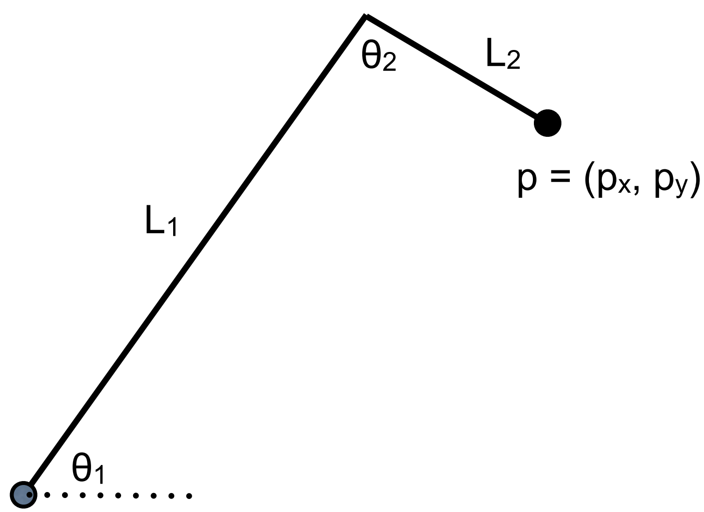
Q: What are the exceptional mechanisms?



Connection to kinematics

A: There are three sets of interest:

b (L_1,L_2,p_x,p_y)	h (c_1,c_2,s_1,s_2)	k at first appearance	Description
4	0	1	generic set
2	1	2	$L_1 = 0$
2	1	2	$L_2 = 0$
0	2	2	$L_1 = L_2 = p_x = p_y = 0$
1	1	3	$L_1 = L_2$ $p_x = p_y = 0$



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Numerical methods for polynomial systems

The key to all of this is that we can encode fiber products in polynomial systems.

Rather than going into details on that, here are a few words about how we solve polynomial systems:

Numerical methods for polynomial systems

Given a polynomial system, there are numerical methods in the field of *numerical algebraic geometry (NAG)* for:

* Computing numerical approximations to all isolated (complex) solutions.

* Computing approximations to generic points (witness points) on positive-dimensional (complex) irreducible components.

* Testing component membership, sampling irreducible components, and various other moves.

* Extracting real subsets of complex solutions sets (new).

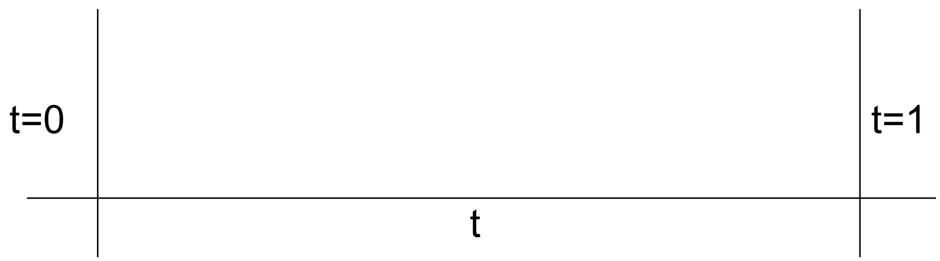
Numerical methods for polynomial systems

Homotopy continuation will provide all isolated (complex) solutions of a polynomial system:

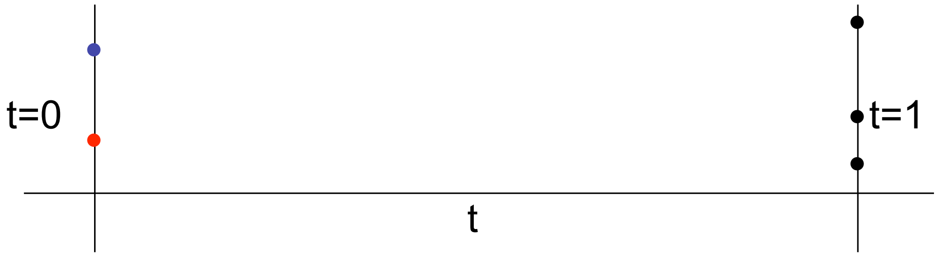
Given polynomial system $f : \mathbb{C}^n \to \mathbb{C}^n$:

1. Choose g:$\mathbb{C}^n \to \mathbb{C}^n$ based on characteristics of f so that it can be solved easily;
2. Form the homotopy H: $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ given by $H(z,t) = f(z) \cdot (1-t) + \gamma g(z) \cdot t$ so that $H(z,1) = g(z)$ and $H(z,0) = f(z)$;
3. Use predictor-corrector methods to track each solution curve to find all isolated complex solutions of $f(z)$. We use **adaptive precision** and follow paths in parallel.

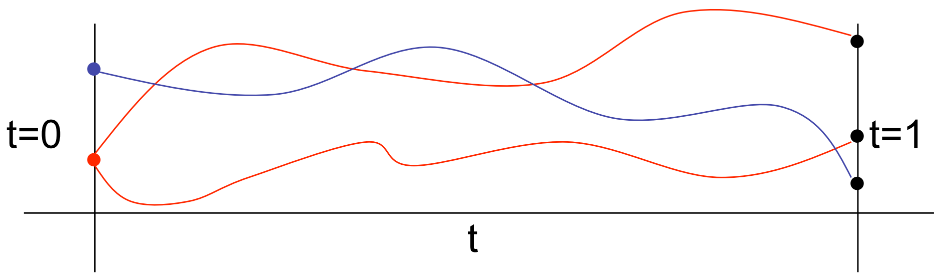
Numerical methods for polynomial systems



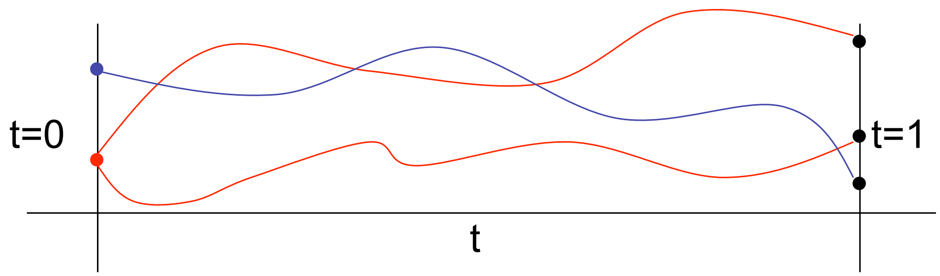
Numerical methods for polynomial systems



Numerical methods for polynomial systems



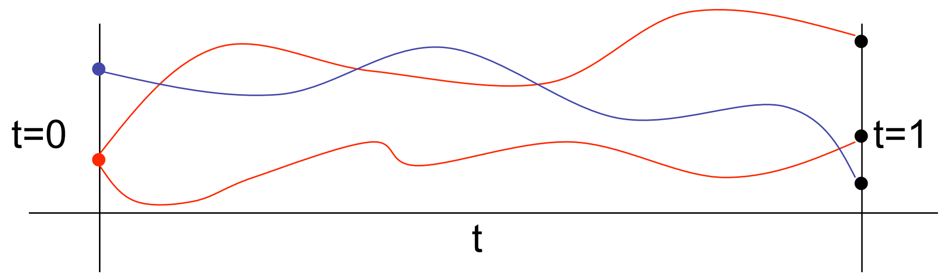
Numerical methods for polynomial systems

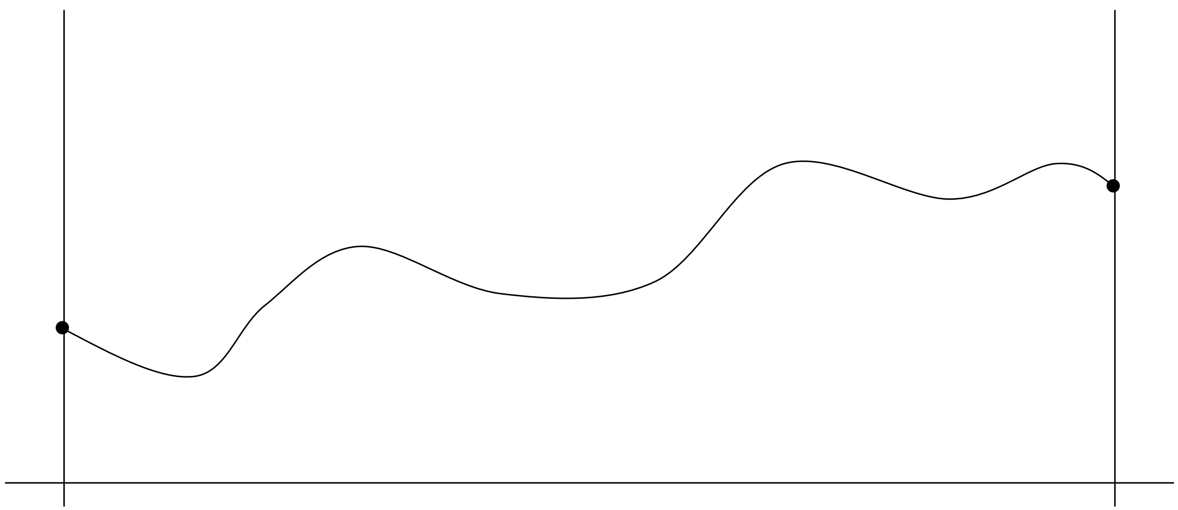


Depending on how you form your start system, many paths could go off to infinity.

Here's how we "track the paths":

Numerical methods for polynomial systems





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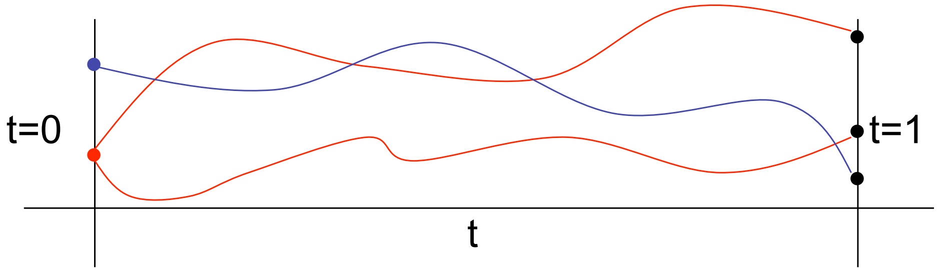

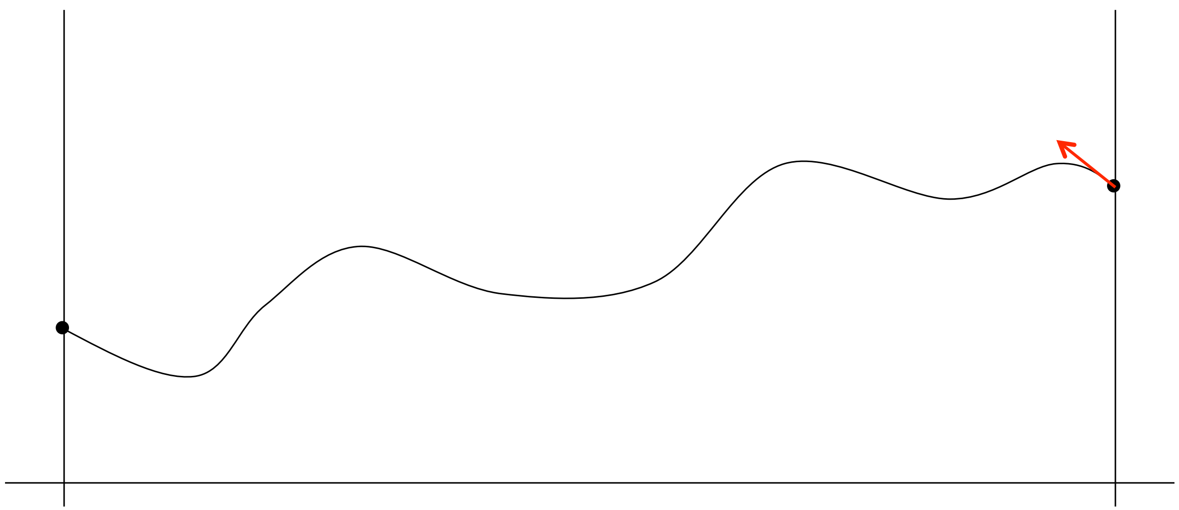

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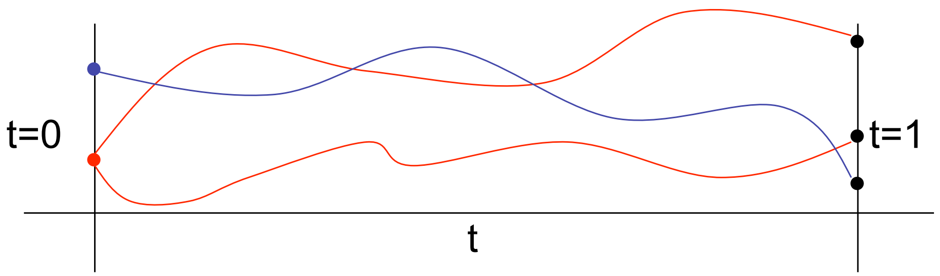
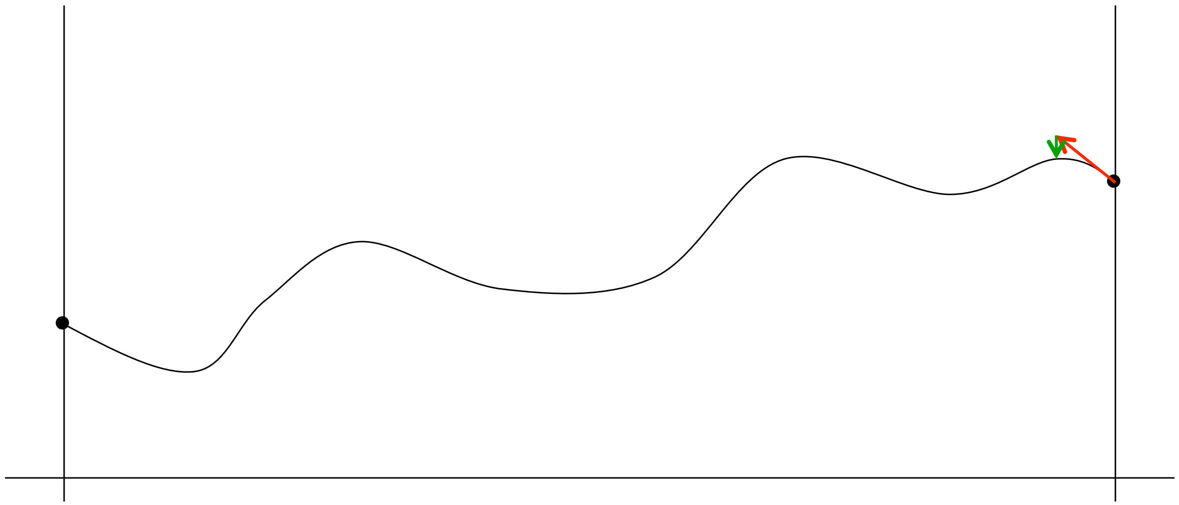
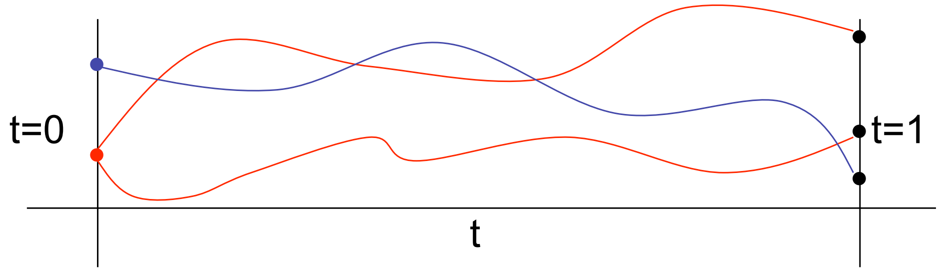
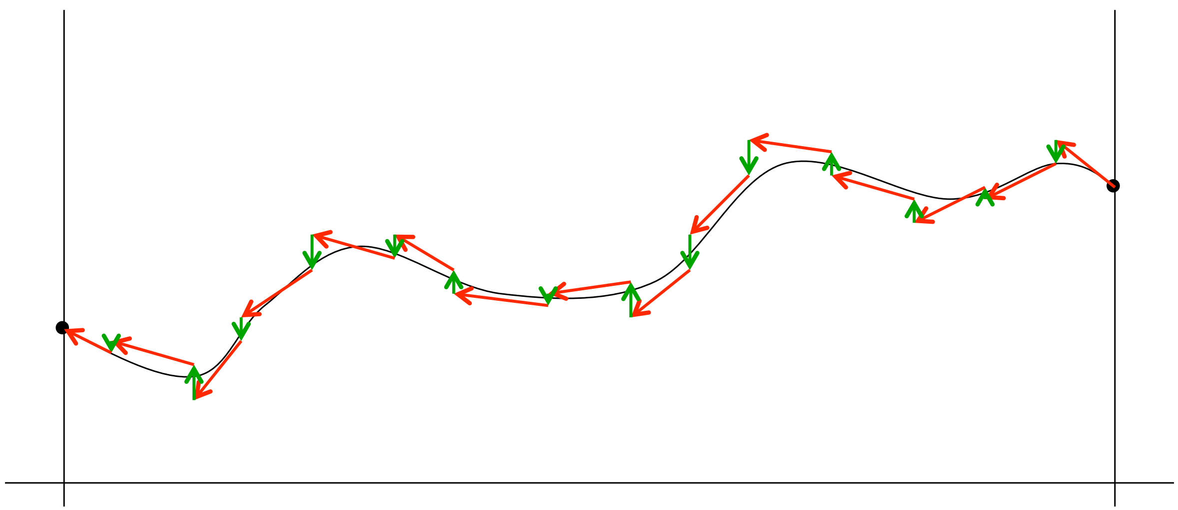


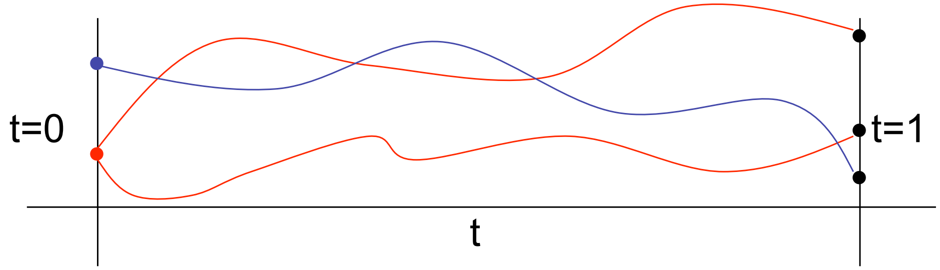


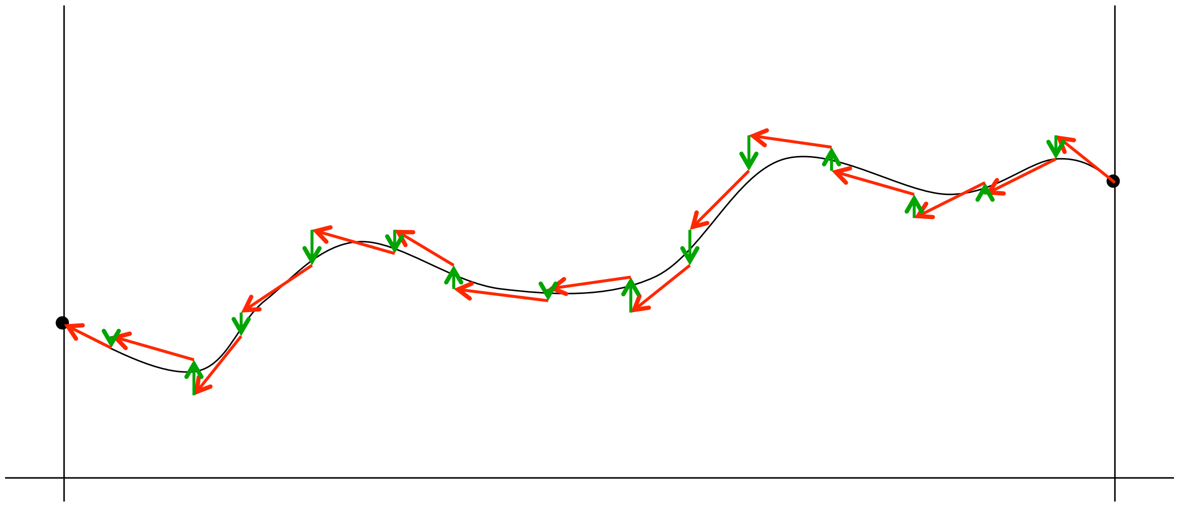
Numerical methods for polynomial systems





Numerical methods for polynomial systems





<u>Not entirely true</u>: Endgames, adaptive precision, parallelization, etc.

Numerical methods for polynomial systems

To find and manipulate positive-dimensional irreducible components, we just take hyperplane sections, i.e., throw in some linear functions with randomly-chosen complex coefficients.

There are several software packages for this (in chronological order):

* PHCpack (Verschelde)
* HOM4PS-2.0 (Lee, Li, Tsai)
* Bertini (next slide)
* NAG4M2 (Leykin, Gross, Rodriguez, Verschelde, B)

* alphaCertified (Hauenstein, Sottile): Certification of solutions.
* Paramotopy (B, Brake, Niemerg): Fast solver for parameterized families of systems.

Bertini

Bertini is free, soon-to-be open source software by:
- Dan Bates, Colorado State University
- Jon Hauenstein, North Carolina State University
- Andrew Sommese, University of Notre Dame
- Charles Wampler, General Motors R&D

Available for download from any of our websites.

Currently executable only, for Linux, Mac, or Windows (Cygwin).

See an upcoming SIAM book for a primer/user's manual.




Polynomial systems for fiber products

Given: Polynomial system f(z;p), describing the motion of the mechanism in question.

The polynomial system for the k^{th} fiber product is:

$$\begin{bmatrix} f(z_1;p) \\ f(z_2;p) \\ \vdots \\ f(z_k;p) \end{bmatrix}$$

We've cooked up various specialized ways to solve these rapidly; see the (upcoming) paper for details.

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Progress and future plans

DONE:
Maple implementation of basic technique
Several basic examples

CURRENT WORK:
Fancy solving (based on regeneration) in Bertini (previous slide)
"Pentad" example - confirming set of exceptional methods

COMING UP:
Slicker ways of solving, as we vary b, h, and/or k
Searching for exceptional mechanisms in other mechanism types
Other applications of fiber products (ideas??)

Progress and future plans

Related work:

M. Husty, A. Karger. Self-motions of Griffis-Duffy type parallel manipulators. *Robotics and Automation*, 2000.

F. Geiss, F. Schreyer. A family of exceptional Stewart-Gough mechanisms of genus 7. *Contemporary Mathematics*, 2009.


SIAM AG13
Colorado State University
July 29-31: Tutorials in the Mountains
August 1-4: Conference

Thanks!