Computational Problems Using Riemann Theta Functions in Sage

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The Riemann Theta Function

Let $\mathcal{H}_g$ denote the space of "Riemann matrices": set of all symmetrix $\Omega \in \mathbb{C}^{g\times g}$ such that $\mathrm{Im}(\Omega)$ is positive definite.

Riemann Theta Function

$\theta : \mathbb{C}^g \times \mathcal{H}_g \to \mathbb{C}$

$$\theta(z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}n\cdot\Omega n + z\cdot n\right)}$$

- Converges absolutely and uniformly on compact sets in $\mathbb{C}^g \times \mathcal{H}^g$.
- Quasiperiodic: integer period. Doubly exponential growth in the columns of Ω. For $m_1, m_2 \in \mathbb{Z}^g$,

$$\theta(z + m_1 + \Omega m_2, \Omega) = e^{-2\pi i\left(\frac{1}{2}m_2\cdot\Omega m_2 + z\cdot m_2\right)}\theta(z,\Omega)$$

The Riemann Theta Function

Slight generalization: Riemann theta functions with characteristic $[\alpha, \beta]$: let $\alpha, \beta \in [0,1)^g$. Define

$$\theta[\alpha,\beta](z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}(n+\alpha)\cdot\Omega(n+\alpha)+(z+\beta)\cdot(n+\alpha)\right)}$$

Addition formulas. One application: rewrite $\theta(z,\Omega)$ in terms of $\theta[\alpha,\beta](0,\Omega)$ for various $\alpha, \beta \in \{0, 1/2\}^g$.

A Special Way to Construct Riemann Matrices

Let $f \in \mathbb{C}[x,y]$, possibly with singularities, with genus g.

1. Desingularize, compactify and determine corresponding Riemann surface Γ.
2. Determine basis for homology $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ and basis for cohomology $\{\omega_1, \ldots, \omega_g\}$. (Basis of holomorphic differentials.)
3. Form the matrices A, B and Ω:

$$\Omega = A^{-1}B \text{ where } (A)_{ij} = \oint_{a_j} \omega_i \text{ and } (B)_{ij} = \oint_{b_j} \omega_i.$$

4. Claim: Ω is a Riemann matrix.

The Schottky Problem

Classifying which Riemann matrices come from algebraic curves:

- Dimension of $g \times g$ Riemann matrices:

$$\frac{g(g+1)}{2}$$

- Dimension of $g \times g$ Riemann matrices derived from algebraic curves:

$$3g - 3$$

This problem was solved by Shiota: Ω is a Riemann matrix derived from an algebraic curve if and only if

$$u(x, y, t) = 2\partial_x^2 \log \theta(\bar{k}x + \bar{l}y + \bar{\omega}t, \Omega)$$

is a solution to the Kadomtsev–Petviashvili Equation.

What is Sage?

Sage is a free open-source mathematics software system licensed under the GPL. It combines the power of many existing open-source packages into a common Python-based interface.
Mission: *Creating a viable free open-source alternative to Magma, Maple, Mathematica, and Matlab.*

Website: `http://www.sagemath.org`

Current Riemann Theta Functionality in Sage

Goal: to provide functionality in Sage for working with Riemann theta functions.

(1) Evaluating Riemann theta functions:
- Given a Riemann matrix, $\Omega \in \mathbb{C}^{g \times g}$, and $z \in \mathbb{C}^g$; compute $\theta(z, \Omega)$ and its derivatives.
  - Arbitrary (user-specified) precision.
  - Sage implementation of the technique of Deconinck, Heil, Bobenko, van Hoeij, and Schmies [1].
- 8-20 times faster than Maple's implementation.
- Code submission needs peer-review: Sage Trac Ticket #6371. (`http://trac.sagemath.org/sage_trac/ticket/6371`)

Current Riemann Theta Functionality in Sage

(2) Given an algebraic curve in $\mathbb{C}[x, y]$, compute a corresponding Riemann matrix. Required components:

- Puiseux series,
- integral basis of $\overline{\mathbb{C}[x]}$ in $\mathbb{C}(x, y)$,
- singularities of a plane algebraic curve: branching numbers, multiplicities, etc.,
- genus,
- monodromy,
- homology basis,
- cohomology basis,
- period matrix.

Implementation

Advertisment: If you have any students, this project is a good way for them to learn these concepts.

Bill Thurston:

"The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical community's standard of valid proofs."

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The Kadomtsev–Petviashvili Equation

Find a solution $u(x,y,t)$ to the non-linear PDE

$$\tfrac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \tfrac{1}{4}(6uu_x + u_{xxx})\right).$$

- Integrable: can be written as the compatibility condition of a Lax-pair.
- Describes 2D shallow water wave-propagation.
- 2D counterpart to the Korteweg – de Vries equation

$$4u_t = 6uu_x + u_{xxx}.$$

Periodic Solutions to KP

KP admits a large family of solutions of the form

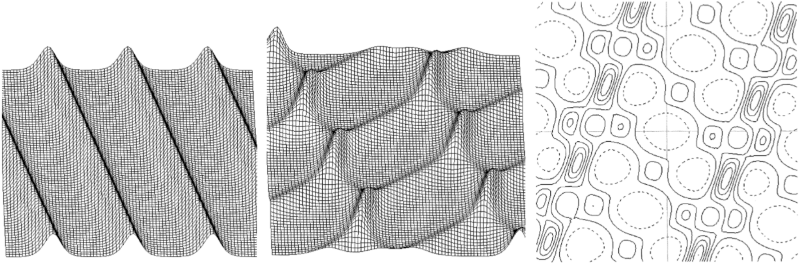
$$u(x,y,t) = 2\partial_x^2 \log \theta(z,\Omega)$$

(up to a constant shift) where θ is the Riemann theta function, the phase variable $z = (z_1, \ldots, z_g)$ is defined as

$$z = \bar{k}x + \bar{l}y + \bar{\omega}t + \bar{\phi}, \quad \bar{k}, \bar{l}, \bar{\omega}, \bar{\phi} \in \mathbb{C}^g$$

and $\Omega \in \mathbb{C}^{g \times g}$ is a Riemann matrix derived from a particular algebraic curve.

- so-called, "genus g solution to KP"
- physically, $\bar{k}$ and $\bar{l}$ are vectors of wave numbers, $\bar{\omega}$ is a vector of frequencies, and $\bar{\phi}$ is a phase shift



Example genus 1,2, and 3 solutions to KP. (Courtesy Dubrovin, Flickinger, and Segur.)

Constructing Genus g Solutions

Theorem

For each Riemann surface Γ of genus g and each point $Q \in \Gamma$ we can construct in a neighborhood of Q a family of solutions to the KP equation. These are parameterized by the non-special divisors of degree g on Γ.

(Very brief) Sketch of construction:

- Pick your favorite Riemann surface and a point Q on that surface.
- Choose a non-special divisor of degree g and construct a corresponding Baker-Akhiezer function ψ from the polynomial, $q(k) = kx + k^2y + k^3t$, where k^{-1} is a local parameter at Q.
- Integrate Abelian differentials of the second kind with double, triple, and quadruple poles at infinity to obtain $\bar{k}, \bar{l}$, and $\bar{\omega}$, respectively.

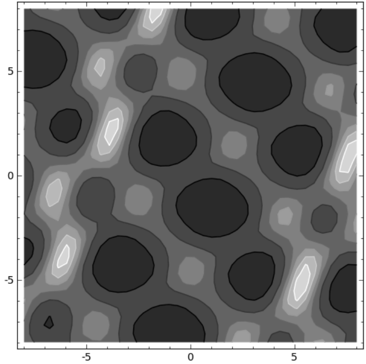
Genus $g = 3$ Solutions

When considering only genus $g \leq 3$ solutions the process for generating solutions is greatly simplified, as the Schottky problem is not an issue. Outline (in the interest of time):

- Substitute $2\partial_x^2 \log \theta(z, \Omega)$ into the KP equation to obtain DE in terms of θ.
- Use Riemann theta addition formulas to rewrite in terms of theta with characteristics (and their derivatives) evaluated at $z = 0$.
    - 8 half-period characteristics $\to$ 8 equations.
- The coefficients of this system are polynomial functions of the components of $\bar{k}, \bar{l}$, and $\bar{\omega}$. Construct a linear system in terms of these coefficients and solve.
- (Time permitting, I will share the details at the end of the talk.)

Sage Implementation

This algorithm is implemented in Sage. It will be part of an "extended examples" documentation for `RiemannTheta`.



Key difference from DFS algorithm: computation of Riemann theta functions.

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Problem

Writing homogenous polynomials as determinants of “Linear Matrix Representations”: “determinantal represenations” of polynomials.

Theorem

Every homogenous polynomial in three variables can be written as

$$f(x,y,z) = \det(Ax + By + Cz)$$

where A, B and C are symmetric matrices.

(Discussions with Daniel Plaumann, Bernd Sturmfels, and Cynthia Vinzant. Additional discussions with Rekha Thomas.)

Applications: Classifying Spectrahedra

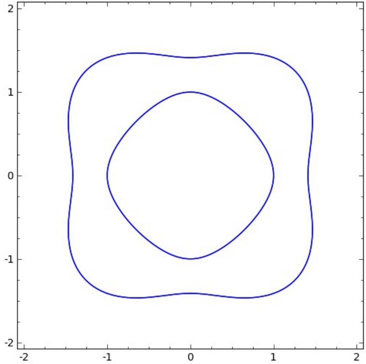
Spectrahedron: the intersection of an affine subspace K with the cone of positive semidefinite matrices S^n_+. Very loosely, *an intersection of finitely many polynomial inequalities.*

- Applications to semidefinite programming.

Claim: all two-dimensional spectrahedra are precisely the subsets of $\mathbb{R}^2$ bounded by rigidly convex algebraic curves, i.e. *Helton-Vinnikov curves*.

Helton-Vinnikov Curves

The algebraic curve has a maximal number of nested ovals. Namely, there are $\lfloor d/2 \rfloor$ nested ovals where $d = \deg f$. The innermost oval bounds a spectrahedron.



Plot in $\mathbb{R}^2$ of the curve $f(x,y) = x^4 + x^2y^2 - 3x^2 + y^4 - 3y^2 + 2$

LMI Construction: The Helton–Vinnikov Theorem

Theorem: (Helton–Vinnikov) Let $f \in \mathbb{R}[x,y,z]_d$ with $f(1,0,0) = 1$ and $\Gamma = \mathcal{V}_{\mathbb{C}}(f) \subset \mathbb{P}^2$. Assume

1. Γ is a non-rational (genus > 0) Helton–Vinnikov curve with the point $(1 : 0 : 0)$ inside its innermost oval.
2. The d real intersection points of Γ with the line $\{z = 0\}$ are distinct non-singular points $Q_1, \ldots, Q_d$ with coordinates $Q_i = (-\beta_j : 1 : 0)$ where $\beta_j \neq 0$.

Then, $f(x,y,z) = \det(\mathsf{Id}_d x + By + Cz)$ where $B = \text{diag}(\beta_1, \ldots, \beta_d)$ and C is real symmetric with diagonal entries

$$c_{ii} = \beta_i \frac{\partial_z f(-\beta_i, 1, 0)}{\partial_y f(-\beta_i, 1, 0)}$$

and off-diagonal entries of C are...

LMI Contruction: The Helton-Vinnikov Theorem

$$c_{jk} = \frac{\beta_k - \beta_j}{\theta[\delta](0)} \frac{\theta[\delta]\left(A(Q_k) - A(Q_j)\right)}{\theta[\epsilon]\left(A(Q_k) - A(Q_j)\right)} \sqrt{\frac{\omega \cdot \nabla\theta[\epsilon](0)}{-d(z/y)}(Q_j)} \sqrt{\frac{\omega \cdot \nabla\theta[\epsilon](0)}{-d(z/y)}(Q_k)}$$

where ϵ is an arbitrary odd theta characteristic, δ is an even theta characteristic such that $\theta[\delta](0) \neq 0$, and $A : \Gamma \to \text{Jac}(\Gamma)$ is the Abel-Jacobi map.

Implementation: Maple

Plaumann wrote a Maple script to compute determinantal representations:

`www.math.uni-konstanz.de/~plaumann/theta.html`

Compute times for the matrix C: (1.6 Ghz dual-core processor, 4GB RAM)

- $d = 4, g = 3$: approx. 5 minutes (longer to plot than compute!)
- $d = 5, g = 6$: approx. 4 hours
- $d = 6, g = 10$: ???

Implementation: Sage

Since the ability to compute Riemann matrices is needed, a Sage implementation is not yet available. Also needed are:

- calculation of the Abel-Jacobi map $A : \Gamma \to \mathrm{Jac}(\Gamma)$,
- calculation of Fay's prime form $E(Q_i, Q_j)$.

Fast calculation of these two functions is a primary goal.

Acknowledgements

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Genus $g = 3$ Solutions: Algorithm

Algorithm

Input: a Riemann matrix Ω and k_1, k_2, l_1;

Output: $k_3, l_2, l_3, \bar{\omega}$ and the corresponding solution

$$u(x, y, t) = 2\partial_x^2 \log \theta(\bar{k}x + \bar{l}y + \bar{\omega}z, \Omega)$$

(Step 1) Choose arbitrary k_1, k_2, and l_1. (Possible due to the Lie Symmetries of KP.)

Genus $g = 3$ Solutions: Algorithm

(Step 2) Construct the 7×7 matrix

$$\begin{pmatrix} \theta_{11}[m_1,0] & \theta_{12}[m_1,0] & \cdots & \theta_{33}[m_1,0] & \theta[m_1,0] \\ \theta_{11}[m_2,0] & & \cdots & & \theta[m_2,0] \\ \vdots & & \cdots & & \vdots \\ \theta_{11}[m_7,0] & & \cdots & & \theta[m_7,0] \end{pmatrix}$$

where...

Genus $g = 3$ Solutions: Algorithm

- $[m_i, 0], m_i \in \{0, \frac{1}{2}\}^3$ are theta characteristics chosen such that the matrix is invertible,
- $\theta_{ij}[m, 0]$ is defined by

$$\theta_{ij}[m, 0] := \frac{\partial^2 \theta[m, 0](0, \Omega)}{\partial z_i \partial z_j},$$

- and $\partial_k^4 \theta[m, 0]$ is defined by

$$\partial_k^4 \theta[m, 0] := \sum_{1 \leq i,j,k,l \leq 3} k_i k_j k_k k_l \frac{\partial^4 \theta[m, 0](0, \Omega)}{\partial z_i \partial z_j \partial z_k \partial z_l}.$$

Genus $g = 3$ Solutions: Algorithm

(Step 3) Compute the inverse matrix

$$\begin{pmatrix} a_{m_1}^{11} & \cdots & a_{m_7}^{11} \\ a_{m_1}^{12} & \cdots & a_{m_7}^{12} \\ \vdots & \cdots & \vdots \\ a_{m_1}^{33} & \cdots & a_{m_7}^{33} \\ a_{m_1} & \cdots & a_{m_7} \end{pmatrix}.$$

(Step 4) Construct the following degree 4 and 6 polynomials:

$$Q_{ij}(k) := -\sum_{m \in \{m_1, \ldots, m_7\}} a_m^{ij} \partial_k^4 \theta[m, 0]$$

$$P_{ij}(k) := \frac{1}{3} \left[k_i^2 Q_{jj}(k) - k_i k_j Q_{ij}(k) + k_j^2 Q_{ii}(k) \right]$$

Genus $g = 3$ Solutions: Algorithm

(Step 5) Finally, find $k_3, l_2, l_3, \omega_1, \omega_2, \omega_3$ by computing the solutions to the following system of six equations

$$k_i l_j - k_j l_i = \sqrt{P_{ij}(k)}, \quad \text{for } 1 \leq i < j \leq 3$$
$$\omega_i = \frac{Q_{ii}(k) - 3l_i^2}{k_i}$$

for some choice of sign on the square roots. Every solution to this system gives rise to a solution to KP.