IMMERSE 2007

Algebra Exercises

5. Week 5

Definition 1. Let R be a ring, and S be another ring with $R \subset S$. An element of S is **integral over** R if it satisfies a monic polynomial with coefficients in R.

Exercise 5.1. (1) Show $(1 + \sqrt{5})/2$ is integral over $\mathbb{Z}$.
(2) Show $(1 + \sqrt{7})/2$ is not integral over $\mathbb{Z}$.

Exercise 5.2. Let $R = k[x, y]/(y^2 - x^3)$. Show y/x is integral over R.

Definition 2. Let R be a ring and I be an ideal of R. An element $r \in R$ is **integral over** I if it satisfies an equation of the form

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0$$

with $a_j \in I^j$, the j^{th} power of I, for each j. Such an equation is called **an equation of integral dependence over** I. The **integral closure of** I **in** R, denoted $\overline{I}$, is the set of all elements which are integral over I. The ideal I is **integrally closed in** R if $I = \overline{I}$.

Exercise 5.3. Show that $\overline{I}$ is closed under taking additive inverse and absorbs products. Show that $I \subset \overline{I}$. Show that if $I \subset J$ then $\overline{I} \subset \overline{J}$.

It is a theorem that $\overline{I}$ is an ideal (closed under addition) and $\overline{\overline{I}} = \overline{I}$. See for example [Huneke–Swanson, Corollary 1.3.1].

Exercise 5.4. In $R = \mathbb{Z}$, find the integral closure of (2), of (4), and of (6).

Exercise 5.5. If R is a domain, show that every radical ideal is integrally closed in R. Show that in a principal ideal domain, every ideal is integrally closed.

Exercise 5.6. Let I be a monomial ideal and let $m = x^\alpha$ be a monomial. Show that $m \in \overline{I}$ if and only if $m^r \in I^r$ for some $r \geq 1$. By Exercise 4.11, $m \in \overline{I}$ if and only if $\alpha \in K(I)$.

Exercise 5.7. Find the integral closure of $I = (x^4, y^3)$ in $k[x, y]$.

For each monomial $m \in \overline{I} \setminus I$, give an explicit equation of integral dependence for m over I.

Exercise 5.8. Find the integral closure of $I = (x^4, y^{11}, x^2y^6)$ in $k[x, y]$.

Exercise 5.9. Let $I, J \subset R = k[x_1, \ldots, x_n]$ be monomial ideals. Show that $\overline{I} = \overline{J}$ if and only if $K(I) = K(J)$. Answer the following true/false questions (giving proofs or counterexamples as appropriate):

a. (i) If $K(I) \subset K(J)$ then $I \subset J$. (ii) If $I \subset J$ then $K(I) \subset K(J)$.
b. (i) If $C(I) \subset C(J)$ then $I \subset J$. (ii) If $I \subset J$ then $C(I) \subset C(J)$.
c. (i) If $I \subset J$ and $K(I) = K(J)$ and I satisfies **NN**, then so does J. (ii) If $I \subset J$ and $K(I) = K(J)$ and J satisfies **NN**, then so does I.

Definition 3. A monomial $m = x_1^{a_1} \ldots x_n^{a_n}$ is called **squarefree** if every $a_i < 2$, that is, m is not divisible by any squares. Let $I \subset R = k[x_1, \ldots, x_n]$ be a monomial ideal. Then I is called **squarefree** if I is generated by a set of squarefree monomials.

Paper Exercise 5.10. a. Show that a monomial ideal I is squarefree if and only if it is radical.

b. Hübl's paper begins with the conjecture that all radical ideals have property **NN**. Has he proved this conjecture for radical monomial ideals? Explain. (Look at Corollary 5.)

Exercise 5.11. In the ring $\mathbb{Z}$,

a. Which ideals are radical?
b. Show: If $a \in \mathbb{Z}$, $a^n \in (2)^{n+1}$ for some $n \geq 1$, then $a \in (4)$.
c. Show: If $a \in \mathbb{Z}$, $p \in \mathbb{Z}$ is prime, and $a^n \in (p)^{n+1}$ for some $n \geq 1$, then $a \in (p^2)$. This holds more generally for any squarefree p.
d. If $a \in \mathbb{Z}$, $a^n \in (4)^{n+1}$ for some $n \geq 1$, then what ideal must a lie in? Give the strongest possible conclusion; prove your conclusion and prove it is the strongest possible, ie, no smaller ideal will do.

Definition 4. Let $I \subset R = k[x_1, \ldots, x_n]$ be an ideal (not necessarily monomial). Let $\mathfrak{m} = (x_1, \ldots, x_d)$. Then I is called **$\mathfrak{m}$-primary** if $I \subseteq \mathfrak{m}$ and I contains a power of $\mathfrak{m}$, that is, $\mathfrak{m}^n \subseteq I$ for some $n \geq 1$.

Exercise 5.12. Show the following.

a. The ideal $\mathfrak{m}^n$ is a monomial ideal. A monomial is in $\mathfrak{m}^n$ if and only if its total degree is equal to or greater than n. In particular, $x_1^n \in \mathfrak{m}^n, \ldots, x_d^n \in \mathfrak{m}^n$.
b. Show $I = (x^3, y^2) \subset R = k[x, y]$ is $\mathfrak{m}$-primary (in this example, $d = 2$).
c. Show that an ideal I is $\mathfrak{m}$-primary if and only if it contains a power of each of the variables; that is, for each $j = 1, \ldots, d$ we have $x_j^{a_j} \in I$ for some $a_j \geq 1$. [HINT: Repeat Exercise 2.16, but don't assume I is monomial.]
d. An ideal I is $\mathfrak{m}$-primary if and only if its radical $\mathrm{rad}(I) = \mathfrak{m}$.
e. Finally, suppose I is a monomial ideal and let S denote the set of monomials in $R \setminus I$. Show that S is a finite set if and only if $\mathrm{rad}(I) = \mathfrak{m}$.

Exercise 5.13. Prove Hübl's Corollary 1. (Use the fact that every face of ∂K is closed, so a face is compact if and only if it is bounded. Show that every bounded face is an F_j; equivalently, every face on a $\partial \Sigma_i$ with $i > S$ is unbounded.)

Exercise 5.14. Finish the proof of Hübl's Corollary 3.

Exercise 5.15. Finish the proof of Hübl's Corollary 6.