IMMERSE 2007

Algebra Exercises

4. Week 4

For the following exercise set, the "Newton polytope (unbounded) of I" is the UNBOUNDED convex hull of all the monomials in I. It is what Hübl denotes $K(I)$.

Exercise 4.1. Show that if I and J are monomial ideals, then so are: $I+J$, $I \cap J$, IJ, and $\operatorname{rad}(I)$.

Exercise 4.2. Let $I = (X^3, Y^2)$. Let $\mathfrak{m} = (X, Y)$. Find the set of monomials that are in I and not in $\mathfrak{m}I$. (That is, the set of monomials in $I \setminus \mathfrak{m}I$.) Repeat for $I = (X^3, X^2Y, Y^2)$.

Exercise 4.3. Let $I \subset R = k[x_1, \dots, x_d]$ be a monomial ideal. Let $\mathfrak{m} = (x_1, \dots, x_d)$. In Exercise 3.6 you showed that there is one and only one irredundant generating set for I consisting of monomials. Show that this irredundant monomial generating set is given by the set of monomials that are in I but not in $\mathfrak{m}I$.

Exercise 4.4. Let I be an ideal (not necessarily monomial). Show that for $a, b \geq 1$, we have $I^a I^b = I^{a+b}$. The zeroth power is defined as $I^0 = (1)$, the unit ideal. Show that $I^0 I^b = I^b = I^b I^0$.

Exercise 4.5. Set $I = (X_0^3 X_1^5, X_0^4 X_1^4, X_0^5 X_1^2) \subseteq k[X_0, X_1]$. Graph the Newton polytope (unbounded) of I in $\mathbb{R}^2$.

Exercise 4.6. Set $I = (X_0^3 X_1^5, X_0^4 X_1^4, X_0^5 X_1^2) \subseteq k[X_0, X_1, X_2]$. Graph the Newton polytope (unbounded) of I in $\mathbb{R}^3$.

Paper Exercise 4.7. For the the monomial ideal $I = (X_0^3 X_1^5, X_0^4 X_1^4, X_0^5 X_1^2) \subseteq k[X_0, X_1, X_2]$, determine the following

(i) C, K, σ_i, s_i, Σ_i, and ∂K, as in Lemma 1.

(ii) S, A_j, $\partial\Sigma_i$, and $A_j \cap \partial\Sigma_i$ as in Lemma 2.

(iii) F_i as in the definition on page 3775.

Exercise 4.8. For this exercise, let V be a vector space over the real numbers $\mathbb{R}$.

Recall the following definition: an **affine subspace** $A \subset V$ is a translation of a vector subspace, $A = L + b$ for some linear subspace (aka vector subspace) $L \subset V$ and some vector $b \in V$.

Show the following:

a. The Minkowski sum of two affine subspaces is again an affine subspace. So is their convex hull.
b. An arbitrary intersection of affine subspaces is again an affine subspace; or empty.
c. Let $S \subset V$ be any nonempty subset. The **affine hull of** S is defined as:

$$\operatorname{aff}(S) = \{v \in V : v = t_1 s_1 + \cdots + t_d s_d, d \geq 1, s_i \in S, t_i \in \mathbb{R}, \sum t_i = 1\}$$

   Show $\operatorname{aff}(S)$ is an affine subspace.
d. Let $S \subset V$ be any nonempty subset. Show that $\operatorname{aff}(S)$ as defined above is equal to the intersection of all the affine subspaces of V containing S.
e. Let $S \subset V$ be any nonempty subset and let $s \in S$. Let $S - s$ be the translation of S by $-s$. Let L be the linear span of $S - s$. Show that $\operatorname{aff}(S) = L + s$.

Exercise 4.9. Recall we defined the dimension of an affine subspace $\dim A$ to be $\dim L$. Show the following.

a. If there is a chain of affine subspaces
$$A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r = A,$$
then $\dim A \geq r$.
b. Furthermore, $\dim A = r$ where r is the largest integer such that there exists a chain of length r, as written above.
c. If $S \subset \mathbb{R}^n$ is a finite set with p elements, then $\dim \operatorname{aff}(S) < p$.
d. We say S is **affinely independent** if S is a finite set with p elements and $\dim \operatorname{aff}(S) = p - 1$. Now, if $A \subset \mathbb{R}^n$ is an affine subspace, show that $\dim A$ is one less than the size of the largest affinely independent subset of A.

Definition 1. Let R be a commutative ring with 1. A **monic polynomial** with coefficients in R is a polynomial whose leading coefficient is 1, that is, a polynomial $f = x^n + r_1 x^{n-1} + \cdots + r_n$.

Exercise 4.10. This exercise is an introduction to the theory of "integral closure" that we will talk about in class soon. For this exercise, let $R = \mathbb{Z}$, the ring of integers with the usual operations.

a. Let $f(x) = x^n + r_1 x^{n-1} + \cdots + r_n$ be a monic polynomial with coefficients $r_i \in \mathbb{Z}$. Suppose $a/b \in \mathbb{Q}$ is a rational number such that $f(a/b) = 0$. Then show $a/b \in \mathbb{Z}$. [HINT: Say a/b is written in lowest terms, ie, a and b have no common factors. Say p is a prime factor of b. Clear denominators in the equation $f(a/b) = 0$, then show p is also a prime factor of a. Now, what is b?]
b. On the other hand, there are lots of irrational real numbers that satisfy monic polynomials with integer coefficients. Show that $\sqrt{2}$ and $\sqrt{2 + \sqrt[5]{3}}$ satisfy this condition. Show that $i = \sqrt{-1}$ satisfies this condition. Show that $\sqrt{2} + \sqrt{3}$ satisfies this condition. [That is, find monic polynomials that these numbers satisfy.]
c. Show that $1/\sqrt{2}$ does NOT satisfy any monic polynomial with integer coefficients.

Exercise 4.11. Show the following.

a. Show $(x^2y^2)^r \in (x^4, x^3y^3, y^4)^r$ for some $r > 0$.
b. Say I is a monomial ideal generated by $m_1, \ldots, m_t$, with corresponding exponent vectors $v_1, \ldots, v_t$. Say n is a monomial with exponent vector w. Suppose $n^r \in I^r$ for some $r > 0$. Then show $w \in K(I)$. [HINT: Write $n^r = g \cdot m_1^{a_1} \ldots m_t^{a_t}$ for some $g \in R$ and $\sum a_i = r$. Show g is a monomial; say its exponent vector is u. Then $rw = u + a_1 v_1 + \cdots + a_t v_t$. Divide by r.]
c. Let I, m_i, v_i, n, and w be as before. Suppose $w \in K(I)$. Then show $n^r \in I^r$ for some $r > 0$. [HINT: Use the fact that we can write $w = c_1 v_1 + \cdots + c_t v_t$ with **rational** c_i, $\sum c_i \geq 1$. Then clear denominators.]
d. For a monomial ideal I and a monomial n, we have $n^r \in I^r$ for some $r > 0$ if and only if $n \in K(I)$.

Definition 2. A polynomial $f \in k[x_1, \ldots, x_n]$ is called **homogeneous of degree** d if every term in f has degree d. For a polynomial f, we write $f = f_0 + f_1 + \cdots + f_n$, where f_d is the sum of all the terms of degree d in f; we say f_d is the **degree** d **part of** f. An ideal $I \subset k[x_1, \ldots, x_n]$ is called a **homogeneous ideal** if there is a generating set for I consisting of homogeneous polynomials.

Exercise 4.12. Let $R = k[x_1, \ldots, x_n]$. Show the following.

a. If f is homogeneous of degree d then $d \cdot f = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}$. Also $f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$.
b. If f is homogeneous of degree d and g is homogeneous of degree e, then fg is homogeneous of degree $d+e$.
c. If I and J are homogeneous ideals, then so are $I + J$, $I \cap J$, and IJ. And so is $\operatorname{rad}(I)$.
d. Show that an ideal I is homogeneous if and only if for every $f \in I$ and for every d, the degree d part $f_d \in I$. [Compare with Exercise 3.5.] Show f is homogeneous if and only if (f) is homogeneous.
e. Let I be a homogeneous ideal. Show that either $I = (1)$ or $I \subseteq (x_1, \ldots, x_n)$.