IMMERSE 2007

Algebra Exercises

2. Week 2

Unless otherwise noted, the letters R and A denote commutative rings with unity, the letters I and J denote ideals, the letter k denotes a field, and letters like X_i denote variables.

Definition 2.1. Let $\sigma : \mathbb{R}^n \to \mathbb{R}$ be a linear function and let s be a real number. The **closed half space** in $\mathbb{R}^n$ corresponding to σ and s is the set $\{v \in \mathbb{R}^n : \sigma(v) \geq s\}$. The **open half space** in $\mathbb{R}^n$ corresponding to σ and s is the set $\{v \in \mathbb{R}^n : \sigma(v) > s\}$.

Exercise 2.1. Let $\sigma : \mathbb{R}^n \to \mathbb{R}$ be a linear function and let s be a real number. Let Σ be the corresponding closed half space and let Σ° be the corresponding open half space.

a. Show that Σ and Σ° are convex.
b. Show that Σ is the topological closure of Σ° and Σ° is the topological interior of Σ.

Paper Exercise 2.2. For the the monomial ideal $I = (X_0^4, X_0^3X_1^3, X_1^4)$ (as on page 3772 of the Hübl paper), determine C, K, σ_i, s_i, and Σ_i, as in Lemma 1.

Exercise 2.3. For each of the sets in Exercise 1.15 that is an ideal, find a finite generating set. Prove that the set actually generates the ideal.

Exercise 2.4. Prove that an ideal I of R is maximal if and only if R/I is a field.

Exercise 2.5. Prove that R is a field if and only if the only ideals of R are (0_R) and R.

Exercise 2.6. Show that if R is a commutative ring, then $R[x]$ is never a field.

Exercise 2.7. a. Say R is a domain. Show that if a polynomial in $R[x]$ is a unit, then it is a nonzero constant.
b. Say $R = k$ is a field. Then the converse is true: if a polynomial in $k[x]$ is a nonzero constant, it is a unit.
c. Show that $(2x + 1)^2 = 1$ in $(\mathbb{Z}/4)[x]$. Conclude that the hypothesis in part (a) that R be a domain cannot be removed in general.

Definition 2.2. Let A be a domain. An element $f \in A$ is called **irreducible** if $f \neq 0$, f is not a unit, and for any factorization $f = gh$, either g or h is a unit.

Exercise 2.8. Let $f \in k[x]$ be a non-constant polynomial in one variable. Prove that the following are equivalent: (1) $k[x]/\langle f\rangle$ is a field. (2) $k[x]/\langle f\rangle$ is an integral domain. (3) f is irreducible.

Exercise 2.9. Let $f = Y^2 - X^3 \in k[X, Y]$. Show f is irreducible. Is $k[X, Y]/\langle f\rangle$ a field, an integral domain, or neither?

Definition 2.3. Let R be a commutative ring with identity and let I be an ideal of R. The **radical** of I is the following set: $\mathrm{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n > 0\}$. Other common notations include $\sqrt{I}$ and $\mathrm{r}(I)$.

Exercise 2.10. Let $R = \mathbb{Z}$. Find the radicals of the ideals: (12), (14), (16), (18).

Exercise 2.11. Prove the following statements:

a. $\mathrm{rad}(I)$ is an ideal of R. [HINT: To show closure under addition, use the binomial theorem.]
b. $I \subseteq \mathrm{rad}(I)$.
c. $\mathrm{rad}(I) = \mathrm{rad}(\mathrm{rad}(I))$.
d. $\mathrm{rad}(IJ) = \mathrm{rad}(I \cap J) = \mathrm{rad}(I) \cap \mathrm{rad}(J)$.

e. $\mathrm{rad}(I) = R$ if and only if $I = R$. [HINT: Use $1 \in R$.]
f. $\mathrm{rad}(I + J) = \mathrm{rad}(\mathrm{rad}(I) + \mathrm{rad}(J))$.
g. $I \subseteq J$ implies $\mathrm{rad}(I) \subseteq \mathrm{rad}(J)$.
h. Suppose $I = (f_1, \ldots, f_m)$. Then $\mathrm{rad}(I) \subseteq \mathrm{rad}(J)$ if and only if for each $i = 1, 2, \ldots, m$ there exists a positive integer n_i such that $f_i^{n_i} \in J$.
i. Suppose $I = (f_1, \ldots, f_s)$ and $J = (g_1, \ldots, g_t)$. Then $\mathrm{rad}(I) = \mathrm{rad}(J)$ if and only if for each $i = 1, 2, \ldots, s$ there exists a positive integer n_i such that $f_i^{n_i} \in J$, and for each $j = 1, 2, \ldots, t$ there exists a positive integer m_j such that $g_j^{m_j} \in I$.
j. Suppose $I \subseteq J$ and that $J = (g_1, \ldots, g_t)$. Then $\mathrm{rad}(I) = \mathrm{rad}(J)$ if and only if for each $j = 1, 2, \ldots, t$ there exists an integer m_j such that $g_j^{m_j} \in I$.

Exercise 2.12. Let I and J be ideals of R.

a. Prove that $I \cap J$ is an ideal.
b. Show by example that the set of products $\{xy : x \in I, y \in J\}$ need not be an ideal.
c. But show that the set of finite sums of products of elements of I and J is an ideal. That is, show the set $\{\sum_{k=1}^{n} x_k y_k : n \geq 0, \text{ and for each } k,\ x_k \in I, y_k \in J\}$ is an ideal.
   It is called the **product ideal of I and J** and denoted IJ.
d. Prove that $IJ \subseteq I \cap J$.
e. Show by example that IJ and $I \cap J$ need not be equal.

Exercise 2.13. Say I is generated by the set S and J is generated by the set T. True or false:

a. $I \cap J$ is generated by $S \cap T$.
b. IJ is generated by $\{st : s \in S, t \in T\}$.
c. $I^2 = II$ is generated by $\{s^2 : s \in S\}$.

(For "true", give a proof; for "false", give a counterexample.)

Paper Exercise 2.14. Let I be the same ideal as in exercise 2.2. Find a generating set for I^2. Repeat exercise 2.2 for I^2.

Let $X_1, \ldots, X_d$ be variables. For any vector of non-negative integers $\alpha \in \mathbb{R}^d$ we may write X^α to denote $X_1^{\alpha_1} \cdots X_d^{\alpha_d}$. Also, we may write $k[X]$ to denote $k[X_1, \ldots, X_d]$, provided it is clear from context whether X denotes a single variable or the collection of variables $X_1, \ldots, X_d$.

Exercise 2.15. Let $X_1, \ldots, X_d$ be variables. We use the notation above, so X denotes the collection of variables. Let $c_\alpha X^\alpha$ be a nonzero monomial, and let $f(X), g(X) \in k[X]$ be polynomials none of whose terms is divisible by $c_\alpha X^\alpha$. Prove that none of the terms of $f(X) - g(X)$ is divisible by $c_\alpha X^\alpha$.

Exercise 2.16. Let $R = k[x_1, \ldots, x_d]$ and $\mathfrak{m} = (x_1, \ldots, x_d)R$. Show the following.

a. If I is a monomial ideal such that $I \neq R$, then $\mathrm{rad}(I) = \mathfrak{m}$ if and only if for each $i = 1, \ldots, d$ there exists an integer $n_i > 0$ such that $x_i^{n_i} \in I$.
b. The ideal $\mathfrak{m}$ is radical, that is, $\mathrm{rad}(\mathfrak{m}) = \mathfrak{m}$.

Exercise 2.17. Let $I_1, \ldots, I_n$ be monomial ideals in $R = k[x_1, \ldots, x_d]$ and set $\mathfrak{m} = (x_1, \ldots, x_d)R$.

a. Prove that the sum $I_1 + \cdots + I_n$ is a monomial ideal.
b. If $\mathrm{rad}(I_1) = \cdots = \mathrm{rad}(I_n) = \mathfrak{m}$, show that $\mathrm{rad}(I_1 + \cdots + I_n) = \mathfrak{m}$.
c. Prove or disprove: If $\mathrm{rad}(I_1 + \cdots + I_n) = \mathfrak{m}$, then $\mathrm{rad}(I_1) = \cdots = \mathrm{rad}(I_n) = \mathfrak{m}$.

Exercise 2.18. Let $I_1, \ldots, I_n$ be monomial ideals in $R = k[x_1, \ldots, x_d]$ and set $\mathfrak{m} = (x_1, \ldots, x_d)R$.

a. Prove that the product $I_1 \cdots I_n$ is a monomial ideal.

b. If $\text{rad}(I_1) = \cdots = \text{rad}(I_n) = \mathfrak{m}$, show that $\text{rad}(I_1 \cdots I_n) = \mathfrak{m}$.
c. Prove or disprove: If $\text{rad}(I_1 \cdots I_n) = \mathfrak{m}$, then $\text{rad}(I_1) = \cdots = \text{rad}(I_n) = \mathfrak{m}$.

Definition 2.4. An **ascending chain** of ideals is an infinite sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$. An ascending chain is said to **stabilize** if there is some N such that for $n \geq N$, $I_n = I_N$. A ring R is called **Noetherian** if every ascending chain of ideals in R stabilizes (the "ascending chain condition").

Exercise 2.19. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals. Then $I = \bigcup I_i = I_1 \cup I_2 \cup \cdots$ is an ideal.

Exercise 2.20. Let A be a Noetherian ring. Show the following.

a. Every field k is Noetherian. [HINT: What ideals are there? What chains are possible?]
b. If $I \subset A$ is an ideal, then A/I is Noetherian.
c. If $I \subset A$ is an ideal, then there is a finite generating set for I. [HINT: Consider the set of all finitely generated ideals contained in I. Use the Noetherian condition plus Zorn's Lemma to show that among all finitely generated ideals contained in I there is a maximal one. Now that must be I, otherwise adding one more element to the generating set of the ideal would give a larger ideal, still finitely generated, violating maximality.]
d. Conversely, if every ideal in R is finitely generated, then R is Noetherian. [HINT: Given a chain, the union of the chain is an ideal and hence finitely generated. The members of the generating set must lie in some ideals in the chain. Take a maximum.]
   Hence: *A ring R is Noetherian if and only if every ideal of R is finitely generated.*
e. (Hilbert's Basis Theorem) The ring $A[X]$ is Noetherian. [HINT: Given an ideal $I \subset A[X]$, show that the set of leading coefficients of polynomials in I forms an ideal in A. By part (c), that ideal is finitely generated. For each of those generators a_i, pick a lowest-degree polynomial f_i with leading coefficient a_i. Show the f_i generate I by considering a lowest-degree member of I outside the ideal generated by the f_i and obtaining a contradiction.]
   Hence: *For a field k, every ideal in a polynomial ring $k[X_1, \ldots, X_d]$ is finitely generated.*
f. The previous part implies that $k[X, Y]$ is Noetherian. Let $R \subset k[X, Y]$ be the subset of polynomials whose terms are of the form either cX^aY^b with $b \leq a\sqrt{2}$ and $c \in k$. (Note that constant terms, $a = b = 0$, meet this condition, so they are allowed.) For example, $X \in R$ but $Y \notin R$.
   Now, show that R is a subring. Then show that R is not Noetherian.
   Thus a subring of a Noetherian ring is not necessarily Noetherian.