IMMERSE 2007

Algebra Exercises

1. Week 1

Unless noted otherwise, the letters R and A denote commutative rings with identity. The letters I and J denote ideals.

Exercise 1.1. Let R be a ring (not necessarily commutative, not necessarily with identity). For any $x \in R$, $x \cdot 0_R = 0_R \cdot x = 0_R$.

Exercise 1.2. Let R be a commutative ring with identity and I an ideal of R.

a. Show that $0_R \in I$.
b. Show that if $a \in I$, then $-a \in I$.
c. Show that if $a, b \in I$, then $a - b \in I$.
d. Show that the set $\{0_R\}$ is an ideal of R.
e. Show that R is an ideal of R.

Exercise 1.3. Let $r \in R$ and set

$$rI = \{ra \mid a \in I\}.$$

Show that rI is an ideal of R.

Exercise 1.4. Prove that the following statements are equivalent:

(i) $I = R$.
(ii) $1_R \in I$.
(iii) I contains a unit.

Exercise 1.5. Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be a collection of ideals of R, where Λ is an index set. (Do not assume that Λ is finite or even countable.) Set $K = \cap_{\alpha\in\Lambda} I_\alpha$. Prove that K is an ideal of R. (Observe that $K \subseteq I_\alpha$ for each $\alpha \in \Lambda$.)

Exercise 1.6. a. Let $f \in R$. Prove that $(f)R$ is an ideal of R and that $f \in (f)R$.
b. Let $f_1, \ldots, f_s$ be elements of R. Prove that $(f_1, \ldots, f_s)R$ is an ideal of R and that $f_1, \ldots, f_s \in (f_1, \ldots, f_s)R$.
c. Let $S \subseteq R$. Prove that $(S)R$ is an ideal of R and that $S \subseteq (S)R$.

Exercise 1.7. Let $S \subseteq R$. Prove that the following statements are equivalent:

(i) $S \subseteq I$.
(ii) $(S)R \subseteq I$.

This fact is useful when you want to show that one ideal is contained in another.

Exercise 1.8. Let $S \subseteq R$. Let $\{I_\alpha\}_{\alpha\in\Lambda}$ denote the collection of ideals of R that contain S. Prove the equality $(S)R = \cap_{\alpha\in\Lambda} I_\alpha$.

Exercise 1.9. Let $S \subseteq R$, and set $I = (S)R$. Prove that I is the unique smallest ideal containing the set S.

Exercise 1.10. a. Give an example to show that $I \cup J$ need not be an ideal.
b. Prove that $I \cup J$ is an ideal of R if and only if either $I \subseteq J$ or $J \subseteq I$.

Exercise 1.11. Let I and J be ideals of R. Set

$$I + J = \{a + b \mid a \in I, b \in J\}.$$

Prove that $I + J$ is an ideal of R and that $I \cup J \subseteq I + J$.

Exercise 1.12. Show that $I + J$ is the unique smallest ideal containing $I \cup J$.

Exercise 1.13. Let $I = (f_1, \ldots, f_n)R$ and let $J = (g_1, \ldots, g_m)R$. Prove that $I + J$ is generated by the set $\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$.

Exercise 1.14. Determine whether the given polynomial in is in the given ideal $I \subseteq \mathbb{R}[x]$.

(1) $f(x) = x^2 - 3x + 2, \qquad I =< x - 2 >$
(2) $f(x) = x^5 - 4x + 1, \qquad I =< x^3 - x^2 + x >$
(3) $f(x) = x^2 - 4x + 4, \qquad I =< x^4 - 6x^2 + 12x - 8, 2x^3 - 10x^2 + 16x - 8 >$
(4) $f(x) = x^3 - 1, \qquad I =< x^9 - 1, x^5 + x^3 - x^2 - 1 >$

Exercise 1.15. Let $R = \mathbb{Z}[x]$. Prove or disprove:

(1) The set K of all constant polynomials in R is an ideal of R.
(2) The set I of all polynomials in R with even constant terms is an ideal of R.
(3) The set I of all polynomials in R with odd constant terms is an ideal of R.

Exercise 1.16. Use Exercise 1.7 to prove the following equalities in the polynomial ring $R = \mathbb{Q}[x, y]$:

a. $(x + y, x - y)R = (x, y)R$.
b. $(x + xy, y + xy, x^2, y^2)R = (x, y)R$.
c. $(2x^2 + 3y^2 - 11, x^2 - y^2 - 3)R = (x^2 - 4, y^2 - 1)R$.

This illustrates that the same ideal can have many different generating sets and that different generating sets may have different numbers of elements.

Exercise 1.17. Let A be a commutative ring with identity. Let f and g be monomials in $R = A[x_1, \ldots, x_d]$. If $(f)R = (g)R$, show that $f = g$.

Exercise 1.18. Let f be a monomial in $R = A[x_1, \ldots, x_d]$ and let n be an integer, $n \geq 1$. Prove that $\deg(f) \leq n$ if and only if there exists a monomial g of degree n such that $g \in (f)R$.

Exercise 1.19. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Exercise 1.20. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Prove that:

(1) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, a_2, \ldots, a_n$ are nilpotent. [HINT: If $b_0 + b_1 x + \cdots + b_m x^m$ is the inverse of f, prove by induction on r that $a_n^{r+1} b_{m-r} = 0$. Hence show that a_n is nilpotent and use the previous exercise.
(2) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent. [HINT: choose a monomial ordering and argue by induction on the number of summands.]
(3) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$. [HINT: Choose a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Then $a_n b_m = 0$, hence $a_n g = 0$ (because $a_n g$ annihilates f and has degree $< m$). Now show by induction that $a_{n-r} g = 0$ $(0 \leq r \leq n)$.]