IMMERSE 2007

Algebra Exercises

6. Supplemental Exercises

Exercise 6.1. This exercise goes back to review the definition of a ring, and explore an object that is "almost a ring". It should be significantly more elementary than most of the other exercises in the whole course.

The **tropical semiring T**, also called the **min-plus algebra**, is defined as follows. As a set, $\mathbf{T} = \mathbb{R} \cup \{\infty\}$. There are two operations, denoted $\oplus$ (tropical addition) and $\odot$ (tropical multiplication), and defined by the following formulas:

$$a \oplus b = \min\{a, b\}, \qquad a \odot b = a + b \text{ (ordinary addition)}$$

Show the following:

a. Find $3 \oplus 7$ and $3 \odot 7$.
b. Show $\oplus$ and $\odot$ are associative.
c. Show $\oplus$ and $\odot$ are commutative.
d. Show $\odot$ distributes over $\oplus$.
e. Is there an identity element for $\oplus$? Are there tropical additive inverses? Explain.
f. Is there an identity element for $\odot$? Are there tropical multiplicative inverses? Explain.
g. In $\mathbb{R}^2$, graph the tropical linear polynomial $y = (3 \odot x) \oplus 7$.
h. In $\mathbb{R}^2$, graph the tropical quadratic polynomial $y = (3 \odot x \odot x) \oplus (7 \odot x) \oplus 12$. The expression $x \odot x$ can be abbreviated $x^{\odot 2}$ or simply x^2.
i. Show the "freshman's dream" holds for all powers in tropical geometry: $(x \oplus y)^{\odot n} = x^{\odot n} \oplus y^{\odot n}$.
j. What is the "characteristic" of $\mathbf{T}$?

Exercise 6.2. Determine all subsets S of $\mathbb{R}^1$ such that both S and its complement are convex; do the same for $\mathbb{R}^2$ and $\mathbb{R}^3$. (For simplicity, you may want to just take S to be closed and its complement open.)

Definition 1. Let R be a commutative ring with identity and let I and J be ideals of R. The **quotient ideal of** I **and** J, or **colon ideal**, is

$$(I :_R J) = \{r \in R \mid rJ \subseteq I\}.$$

It is often written simply $(I : J)$ if the ring R is clear from the context.

Exercise 6.3. Let I, J, K be ideals of R and let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a collection of ideals of R. Prove the following statements:

a. $(I : J)$ is an ideal of R.
b. $(I : J) = R$ if and only if $J \subseteq I$.
c. $I \subseteq (I : J)$.
d. $(I : J)\, J \subseteq I$.
e. $((I : J) : K) = (I : JK) = ((I : K) : J)$.
f. $\left(\bigcap_{\lambda \in \Lambda} I_\lambda : J\right) = \bigcap_{\lambda \in \Lambda} (I_\lambda : J)$.
g. $\left(J : \sum_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} (J : I_\lambda)$. [HINT: Use the fact that $\sum I_\lambda$ is the ideal generated by $\bigcup I_\lambda$, which you proved in Exercise 1.11 and Exercise 1.12.]

Exercise 6.4. Let I be a radical ideal and J an arbitrary ideal. Prove that $(I : J)$ is radical.

Exercise 6.5. Let I be an integrally closed ideal and J an arbitrary ideal. Prove that $(I : J)$ is integrally closed.

Exercise 6.6. Let I and J be monomial ideals. Show that $(I : J)$ is a monomial ideal. Also, show that if I and J are homogeneous ideals, so is $(I : J)$.

Exercise 6.7. Let I and J be ideals in R. Show the following.

a. We have $((I : J^a) : J) = \left(I : J^{a+1}\right)$.
b. If $a \leq b$ then $(I : J^a) \subseteq \left(I : J^b\right)$. By the previous part, if we keep coloning by J, the ideal just keeps growing.
c. The **saturation of** I **with respect to** J,

$$(I : J^\infty) = \bigcup_{a \geq 0} (I : J^a),$$

is an ideal containing I, and such that $((I : J^\infty) : J^a) = (I : J^\infty)$ for all a. (This justifies the name "saturated": putting more Js doesn't change it any more.)
d. If I and J are monomial, so is $(I : J^\infty)$.
e. If I and J are homogeneous, so is $(I : J^\infty)$.
f. If I is a homogeneous ideal, we define the **saturation of** I to be $I^{\text{sat}} = (I : \mathfrak{m}^\infty)$. Show this is a homogeneous ideal containing I and that $(I^{\text{sat}})^{\text{sat}} = I^{\text{sat}}$.
g. If $I \subset J$ are homogeneous then $I^{\text{sat}} \subset J^{\text{sat}}$.
h. For I homogeneous we have $I \subset I^{\text{sat}} \subset \text{rad}(I)$. Show a radical homogeneous ideal is saturated.
i. For I homogeneous, I^{sat} is the unique smallest saturated ideal containing I. (A homogeneous ideal is saturated if it equals its saturation. Here, "smallest" means I^{sat} is contained in any saturated ideal containing I.)
j. Let I be homogeneous and $d \geq 0$. The **degree** d **piece of** I, denoted I_d, is the set of homogeneous forms of degree d in I (together with 0). Show that each I_d is a vector space over k, contained in $\mathfrak{m}_d$. Show that $\mathfrak{m}_d$ is finite dimensional, and hence so is I_d.
k. For I homogeneous, $I_d = (I^{\text{sat}})_d$ for $d \gg 0$. [HINT: Use the Noetherian property. It says I^{sat} is finitely generated. So there is a maximum degree a of the generators, and a b such that all the generators multiply $\mathfrak{m}^b$ into I (taking the maximum of the b's of the various generators). Show that $I_d = (I^{\text{sat}})_d$ for $d \geq a+b$.]
l. For I and J homogeneous, $I^{\text{sat}} = J^{\text{sat}}$ if and only if $I_d = J_d$ for $d \gg 0$.

Exercise 6.8. Let $I_1 \subset I_2 \subset \ldots$ be an ascending chain of ideals and $J = \bigcup I_i$ be their union. Show the following.

a. If each I_i is prime, so is J.
b. If each I_i is radical, so is J.
c. If each I_i is homogeneous (in a polynomial ring), so is J. Ditto, monomial. Ditto, homogeneous and saturated.
d. (Harder): If each I_i is integrally closed, so is J.

Exercise 6.9. Let A be any commutative ring with 1 (not necessarily Noetherian) and let $R = A[x_1, \ldots, x_n]$. Let $I \subset R$ be an ideal generated by a set of monomials. (Here, a monomial is a polynomial with one term and coefficient 1.) Then show I is generated by a finite set of monomials. [HINT: Imitate the proof given in class that the lexicographic order is a well-ordering.]

Thus, even though R is not Noetherian if A is not, at least all ideals generated by monomials are actually finitely generated by monomials.

Exercise 6.10. Prove the statement in Hübl's Remark 4, (ii): that $K = F + \mathbb{R}^{d+1}_{\geq 0}$.

Challenge Exercise 6.11. Let $I \subset R = k[x_1, \dots, x_n]$ be a monomial ideal and $\mathfrak{m} = (x_1, \dots, x_n)$. Prove or disprove the following.

a. If I satisfies **NN**, then so does I^2. (What about higher powers?)
b. If I satisfies **NN**, then so does $\mathfrak{m}I$.
c. If I^2 satisfies **NN**, then so does I.
d. If $\mathfrak{m}I$ satisfies **NN**, then so does I.

If possible, salvage those which are false—that is, change the statement so it becomes true. For example, strengthen the hypothesis or weaken the conclusion. (I have no idea if these are true or false; trivial or impossible.)

Challenge Exercise 6.12. For those of you who like polyhedra: Define property **NN** for polyhedra. You may want to restrict to bounded polyhedra (i.e., polytopes), or lattice polytopes (i.e., convex hull of points in $\mathbb{Z}^n$), or polyhedra P such that $P + v \subseteq P$ if and only if $v \in \mathbb{R}^n_{\geq 0}$, or possibly other reasonable restrictions. Once you have a good definition of property **NN**, rewrite Hübl's paper, proving all the analogous theorems for polyhedra. Write and publish it collaboratively with your IMMERSE classmates and/or teachers.

Challenge Exercise 6.13. For those of you who like algebra: Study $I^{p/q} = \{r : r^q \in I^p\}$ and $\tilde{I}^{p/q} = \bigcup\{I^{r/s} : \frac{r}{s} = \frac{p}{q}\}$. Relate them to the better-known ideals $\overline{I}^{p/q} = \{r : r^q \in \overline{I^p}\}$. Write and publish your results collaboratively with your IMMERSE classmates and/or teachers. (Warning: Results are not guaranteed. This may lead nowhere.)

Challenge Exercise 6.14. Generalize Hübl's Corollary 6 to higher dimensions, that is, more than 2 variables.

Suggestion: In $\mathbb{R}^2$, a line $ax + by = c$ with $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$ has slope equal to an integer or the reciprocal of an integer if and only if $a = 1$ or $b = 1$. Perhaps rephrasing Corollary 6 using this idea would allow for a nice generalization.