Cohomology Tables of Coherent Sheaves

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Overview

Graded Betti numbers

The coefficients of the Hilbert polynomial are the fundamental numerical invariants of a graded S-module.

The graded Betti numbers β_{ij} of a minimal resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_{n+1} \leftarrow 0$$

are finer numerical invariants!

Canonical curves of genus 7 [Schreyer 1986]

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1	–	–	–	–	–
–	10	20	15	4	–
–	4	15	20	10	–
–	–	–	–	–	1

1	–	–	–	–	–
–	10	16	9	–	–
–	–	9	16	10	–
–	–	–	–	–	1

1	–	–	–	–	–
–	10	16	3	–	–
–	–	3	16	10	–
–	–	–	–	–	1

1	–	–	–	–	–
–	10	16	–	–	–
–	–	–	16	10	–
–	–	–	–	–	1

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1	–	–	–	–	–
–	10	20	15	4	–
–	4	15	20	10	–
–	–	–	–	–	1

trigonal

1	–	–	–	–	–
–	10	16	9	–	–
–	–	9	16	10	–
–	–	–	–	–	1

$\exists\ g^2_6$

1	–	–	–	–	–
–	10	16	3	–	–
–	–	3	16	10	–
–	–	–	–	–	1

4-gonal

1	–	–	–	–	–
–	10	16	–	–	–
–	–	–	16	10	–
–	–	–	–	–	1

general case, $char(\mathbb{K}) \neq 2$

Attempting to prove the Multiplicity Conjectures of Herzog, Huneke and Srinivasan, Boij and Söderberg [2007] made a big step forward towards an answer of the fundamental question:

Which Betti tables are possible?

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Let us think of a Betti table $\beta(M) = (\beta_{ij}(M))$ as an element of the vector space

$$\bigoplus_{j \in \mathbb{Z}} \mathbb{Q}^{n+2}.$$

Since $\beta(M \oplus N) = \beta(M) + \beta(N)$, it is natural to consider the convex cone spanned by all possible Betti tables.

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The Boij-Söderberg conjectures describe this cone completely!

Pure Resolutions

A pure resolution is the resolution of a CM-Module, which has shape

$$0 \leftarrow M \leftarrow S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \ldots \leftarrow S(-d_c)^{\beta_c} \leftarrow 0$$

Proposition

The Betti numbers $\beta_i = \beta_{i,d_i}$ of a pure resolution are determined by the degree sequence

$$(d_0, d_1, \ldots, d_c)$$

up to a common factor.

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Proof: The numerator of the Hilbert series $\sum_{i=0}^{c}(-1)^i \beta_i z^{d_i}$ vanishes to order c at $z = 1$. This gives c equations for $c + 1$ Betti numbers $\beta_0, \ldots, \beta_c$. □

Know any modules with these resolutions?

The following Betti tables belong to the degree sequences

$$(0, 2, 3, 5, 6, 8) \qquad \text{and} \qquad (0, 2, 3, 4, 6, 8)$$

1	–	–	–	–	–
–	10	16	–	–	–
–	–	–	16	10	–
–	–	–	–	–	1

5	–	–	–	–	–
–	60	128	90	–	–
–	–	–	–	20	–
–	–	–	–	–	3

respectively.

The Boij-Söderberg Conjectures [2007]

Now Theorems (– and Schreyer, JAMS, 2009)

1. Existence. For every degree sequence there exists a CM-module with a pure resolution.
2. Spanning. The cone of Betti tables is generated by Betti tables of pure resolutions.
3. Decomposition. Each Betti table is a unique positive rational linear combination of pure Betti tables in a unique chain of degree sequences.

Here ”chain” refers to the natural partial order of degree sequences

$$(d_0, d_1, \dots, d_c) \leq (e_0, e_1, \dots, e_c) \Leftrightarrow d_i \leq e_i.$$

General Modules, and the Multiplicity Conjecture

Theorem (Boij-Söderberg, 2008, the non-CM case)

The cone of Betti tables of arbitrary modules is generated by Betti tables of pure complexes of CM-modules of various codimensions.

Corollary

The Multiplicity Conjectures of Herzog, Huneke and Srinivasan (and more) are true!

Cohomology Tables

Let $\mathcal{E}$ be a coherent sheaf on $\mathbb{P}^n$, for example a vector bundle. We have the dimensions of the cohomology groups

$$\gamma_{ij} = h^i(\mathbb{P}^n, \mathcal{E}(j)).$$

We identify the cohomology table $\gamma(\mathcal{E}) = (\gamma_{ij})$ with an element of

$$\prod_{j \in \mathbb{Z}} \mathbb{Q}^{n+1}.$$

Supernatural Sheaves

A sheaf $\mathcal{E}$ on $\mathbb{P}^m$ has natural cohomology, if for each twist k at most one group $H^i(\mathcal{E}(k)) \neq 0$. It is supernatural, if in addition the Hilbert polynomial

$$\chi(\mathcal{E}(t)) = \frac{rank\ \mathcal{E}}{n!} \prod_{k=1}^{n} (t - z_k)$$

has pairwise distinct integral roots $z = (z_1 > \ldots > z_n)$.
(Here n is the dimension of the support of the sheaf.)
We denote the cohomology table of a supernatural sheaf with root sequence z and degree $n!$ by γ^z

Existence

Theorem
There exists supernatural sheaf bundle for any given zero sequence $z = (z_1, \ldots, z_s)$.

Example

The Cohomology table of a supernatural rank 3 vector bundle on $\mathbb{P}^3$ with roots $z = (3, -1, -4)$ is

...	90	45	16	0	0	0	0	0	0	0	...	3
...	0	0	0	6	5	0	0	0	0	0	...	2
...	0	0	0	0	0	6	10	9	0	0	...	1
...	0	0	0	0	0	0	0	0	20	54	...	0
	-4			-1				3				$k \backslash i$

Here the entry in position (k, i) is the dimension of the cohomology group $H^i(\mathcal{E}(k - i))$.

Some Pairings

Main idea: consider the pairing

$$\langle \beta, \gamma \rangle = \sum_{\{i,j,k \mid j \leq i\}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k}.$$

We abbreviate

$$\langle M, \mathcal{E} \rangle = \langle F, \mathcal{E} \rangle = \langle \beta(F), \gamma(\mathcal{E}) \rangle$$

Positivity Theorem 1

Theorem

Let F be any free resolution of an S-module, and let $\mathcal{E}$ be any coherent sheaf. Then

$$\langle F, \mathcal{E} \rangle \geq 0$$

Moreover, if

$$\begin{aligned} 0 &> \text{reg}\, M + \text{reg}\, \mathcal{E}, \text{ and} \\ 0 &> \text{reg}\, F_{i-1} + \text{reg}\, \oplus_k H^i \mathcal{E}(k) \text{ for all } i > 0, \end{aligned}$$

then $\langle F, \mathcal{E} \rangle = 0$.

Truncation

We modify $\langle -, \mathcal{E} \rangle$ by a suitable truncation

$$\begin{aligned} \langle \beta, \gamma \rangle_{c,\tau} &= \sum_{\{i,j,k | j \leq i-2 \text{ or } j<\tau\}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k} \\ &- \sum_{\{i,j,k | j=i-1=\tau, k \leq c+1\}} \beta_{\tau+1,k} \gamma_{\tau,-k} \\ &+ \sum_{\{i,j,k | j=i=\tau, k \leq c\}} \beta_{\tau,k} \gamma_{\tau,-k} \end{aligned}$$

Theorem

The functional $\langle -, \mathcal{E} \rangle_{\tau,c}$ is non negative on minimal free resolutions.

An analogue of Boij-Söderberg for vector bundles

Theorem (– and Schreyer)

The cohomology table of an arbitrary vector bundle on $\mathbb{P}^n$ is a finite positive linear combination of cohomology tables of supernatural bundles.

Boij-Söderberg analog for coherent sheaf

If Z is an infinite set of zero sequences, $(q_z)_{z\in Z}$ a sequence of numbers, and γ is a cohomology table, we write $\gamma = \sum_{z\in Z} q_z\gamma^z$, to mean that each entry $\sum_{z\in Z} q_z\gamma^z_{i,d}$ converges to $\gamma_{i,d}$.

Theorem (E-S, 2009)

Let $\gamma(\mathcal{F})$ be the cohomology table of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$. There is a unique chain of zero-sequences Z and a unique expression

$$\gamma(\mathcal{F}) = \sum_{z\in Z} q_z\gamma^z,$$

where the q_z are positive numbers.

Example

The ideal sheaf $\mathcal{I}_p$ of a point in $\mathbb{P}^2$ has the cohomology table

$\cdots$	10	6	3	1							2
$\cdots$	1	1	1	1	1						1
						2	5	9	14	$\cdots$	0
$\cdots$	-4	-3	-2	-1	0	1	2	3	4	$\cdots$	$d \backslash i$

where we dropped zero entries for the better visibility of the shape. Then

$$\gamma(\mathcal{I}_p) = \sum_{k=2}^{\infty} q_{(0,-k)} \gamma^{(0,-k)}$$

with

$$q_{(0,-k)} = \frac{2}{(k-1)k(k+1)}.$$

Idea of proof

Look at the supernatural sheaf with largest zero-sequence with the same upper shape as the given sheaf,

···	10	6	3	1							2
···	1	1	1	1	1						1
						2	5	9	14	···	0

Idea of proof

Look at the supernatural sheaf with largest zero-sequence with the same upper shape as the given sheaf,

⋯	10	6	3	1							2
⋯	1	1	1	1	1						1
						2	5	9	14	⋯	0

in our case $\gamma^{(0,-2)}$,

⋯	24	15	8	3							2
					1						1
						3	8	15	24	⋯	0

Idea of proof

Look at the supernatural sheaf with largest zero-sequence with the same upper shape as the given sheaf,

⋯	10	6	3	1							2
⋯	1	1	1	1	1						1
						2	5	9	14	⋯	0

in our case $\gamma^{(0,-2)}$, and subtract as much as possible,

⋯	24	15	8	3							2
					1						1
						3	8	15	24	⋯	0

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such that corners stay non-negative:

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$\cdots$	1	1	1	1	1						1
						2	5	9	14	$\cdots$	0

in our case $\gamma^{(0,-2)}$, and subtract as much as possible,

$\cdots$	24	15	8	3							2
					1						1
						3	8	15	24	$\cdots$	0

such that corners stay non-negative: $\gamma - \frac{1}{3}\gamma^{(0,-2)}$

$\cdots$	2	1	$\frac{1}{3}$	0							2
$\cdots$	1	1	1	1	$\frac{2}{3}$						1
						1	$\frac{7}{3}$	4	6	$\cdots$	0

Idea of proof, 2nd step

Now look at

$\cdots$	2	1	$\frac{1}{3}$								2
$\cdots$	1	1	1	1	$\frac{2}{3}$						1
						1	$\frac{7}{3}$	4	6	$\cdots$	0

subtract a multiple of $\gamma^{(0,-3)}$:

$\cdots$	18	10	4								2
$\cdots$				2	2						1
						4	10	18	28	$\cdots$	0

Idea of proof, 2nd step

Now look at

$$
\begin{array}{ccccccccccc|c}
\cdots & 2 & 1 & \frac{1}{3} & & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & \frac{2}{3} & & & & & & 1 \\
 & & & & & & 1 & \frac{7}{3} & 4 & 6 & \cdots & 0 \\
\hline
\end{array}
$$

subtract a multiple of $\gamma^{(0,-3)}$:

$$
\begin{array}{ccccccccccc|c}
\cdots & 18 & 10 & 4 & & & & & & & & 2 \\
\cdots & & & & 2 & 2 & & & & & & 1 \\
 & & & & & & 4 & 10 & 18 & 28 & \cdots & 0 \\
\hline
\end{array}
$$

We get $\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)}$.

$$
\begin{array}{ccccccccccc|c}
\cdots & \frac{1}{2} & \frac{1}{6} & & & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & \frac{5}{6} & \frac{1}{2} & & & & & & 1 \\
 & & & & & & \frac{2}{3} & \frac{3}{2} & \frac{5}{2} & \frac{11}{3} & \cdots & 0 \\
\hline
\end{array}
$$

Idea of proof, 2nd step

Now look at

$\cdots$	2	1	$\frac{1}{3}$								2
$\cdots$	1	1	1	1	$\frac{2}{3}$						1
						1	$\frac{7}{3}$	4	6	$\cdots$	0

subtract a multiple of $\gamma^{(0,-3)}$:

$\cdots$	18	10	4								2
$\cdots$				2	2						1
						4	10	18	28	$\cdots$	0

We get $\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)}$. Continue ... !

$\cdots$	$\frac{1}{2}$	$\frac{1}{6}$									2
$\cdots$	1	1	1	$\frac{5}{6}$	$\frac{1}{2}$						1
						$\frac{2}{3}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{11}{3}$	$\cdots$	0

Idea of proof

$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)}$$

$$\begin{array}{ccccccccccc|c}
\cdots & \frac{1}{10} & & & & & & & & & & 2 \\
\cdots & 1 & 1 & \frac{9}{10} & \frac{7}{10} & \frac{2}{5} & & & & & & 1 \\
 & & & & & & \frac{1}{2} & \frac{11}{10} & \frac{9}{5} & \frac{13}{5} & \cdots & 0 \\
\hline
\end{array}$$

Idea of proof

$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)}$$

$\cdots$	$\frac{1}{10}$										2
$\cdots$	1	1	$\frac{9}{10}$	$\frac{7}{10}$	$\frac{2}{5}$						1
						$\frac{1}{2}$	$\frac{11}{10}$	$\frac{9}{5}$	$\frac{13}{5}$	$\cdots$	0

Proposition (Key claim)

All entries of the table stay non-negative through out this process.

Idea of proof

$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)} - \ldots - \frac{1}{168}\gamma^{(0,-7)}$$

$\cdots$	$\frac{1}{28}$													2
$\cdots$	1	1	$\frac{27}{28}$	$\frac{25}{28}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{13}{28}$	$\frac{1}{4}$						1
									$\frac{2}{7}$	$\frac{17}{28}$	$\frac{27}{28}$	$\frac{19}{14}$	$\cdots$	0

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$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)} - \ldots - \frac{1}{168}\gamma^{(0,-7)}$$

$\ldots$	$\frac{1}{28}$														2
$\ldots$	1	1	$\frac{27}{28}$	$\frac{25}{28}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{13}{28}$	$\frac{1}{4}$							1
									$\frac{2}{7}$	$\frac{17}{28}$	$\frac{27}{28}$	$\frac{19}{14}$	$\ldots$		0

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Thanks!