How to Draw Tropical Planes[1]

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[1] Joint work with Sven Hermann, Anders Jensen and Michael Joswig

Ideals, Varieties and Algorithms (in the Tropics)

Fix field K with a valuation, such as $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{Q}(t)$, $\mathbb{C}$, $\mathbb{C}_p$, $\mathbb{C}\{\{t\}\}$. If $f \in K[x_1, \ldots, x_n]$ and $w \in \mathbb{R}^n$ then the *initial form* $\mathrm{in}_w(f)$ is the sum of all terms in the expansion of f that have maximal w-weight. The *tropical hypersurface* of f is the $(n-1)$-dimensional complex

$$\mathcal{T}(f) \;=\; \{\, w \in \mathbb{R}^n \,:\, \mathrm{in}_w(f) \text{ is not a monomial} \,\}.$$

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A *tropical prevariety* is a finite intersection of such hypersurfaces.
Every ideal I in $K[x_1, \ldots, x_n]$ defines a *tropical variety* as follows:

$$\mathcal{T}(I) \;=\; \bigcap_{f \in I} \mathcal{T}(f).$$

If I is generated by linear forms then $\mathcal{T}(I)$ is a *tropicalized plane*.

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Facts: Every tropical variety is a prevariety (but not vice versa), i.e. every ideal I has a finite *tropical basis*. If I is prime of dim d then $\mathcal{T}(I)$ is a strongly connected pure polyhedral complex of dimension d in $\mathbb{R}^n$. **You can compute $\mathcal{T}(I)$ with GFan** !!

The Grassmannian and the Dressian

The *Plücker ideal* $I_{d,n}$ is a prime ideal in a polynomial ring in $\binom{n}{d}$ variables (over $\mathbb{Q}$). Its elements are algebraic relations among the $d \times d$-minors of a $d \times n$-matrix. It is generated by quadrics such as

$$x_{12}\, x_{34} \;-\; x_{13}\, x_{24} \;+\; x_{14}\, x_{23} \qquad\qquad (\text{for } d = 2, n = 4).$$

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The *Grassmannian* $\mathrm{Gr}(d,n)$ is the tropical variety $\mathcal{T}(I_{d,n})$. This is a pure fan of dimension $d(n{-}d)+1$. We represent this fan as a polyhedral complex of dimension $d(n{-}d) - n$. The *Dressian* $\mathrm{Dr}(d,n)$ is the prevariety defined by the set of quadrics in $I_{d,n}$.

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Theorem (Speyer-St.)

The Grassmannian $\mathrm{Gr}(d,n)$ *is the parameter space for all tropicalized* $(d{-}1)$*-planes in* $\mathbb{TP}^{n-1}$*. The Dressian* $\mathrm{Dr}(d,n)$ *is the parameter space for all tropical* $(d{-}1)$*-planes in* $\mathbb{TP}^{n-1}$*.*

The Grassmannian of Planes in $\mathbb{TP}^5 = \mathbb{R}^6/\mathbb{R}(1,1,1,1,1,1)$

In 2003, the dark days before `GFan`, David Speyer and I computed the Grassmannian $\mathrm{Gr}(3,6)$. We found that $\mathrm{Gr}(3,6)$ is a three-dimensional simplicial complex with 65 vertices, 550 edges, 1395 triangles and 1035 tetrahedra. This complex triangulates $\mathrm{Dr}(3,6)$.

Its homology is that of a bouquet of 126 3-spheres (cf. [Hacking]).

There are 1035 generic tropical planes in $\mathbb{TP}^5$. Up to symmetry there are seven types. Each plane is a contractible complex which we think of as a "two-dimensional tree on six taxa".

Question: How to draw a tropical plane ?

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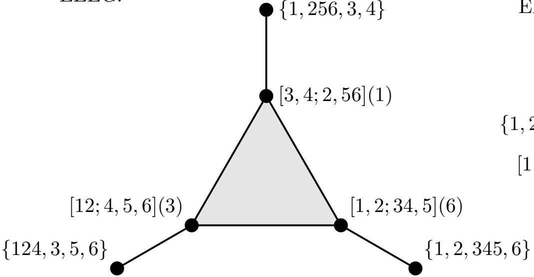
Question: How to draw a tropical plane ?

Answer: Draw the bounded part or draw the unbounded part.

The bounded part is a cell complex whose vertices are labeled by rank 3 matroids on $\{1,2,3,4,5,6\}$. This picture is dual to a *matroid subdivision* of the hypersimplex $\Delta(3,6)$.

EEEG:



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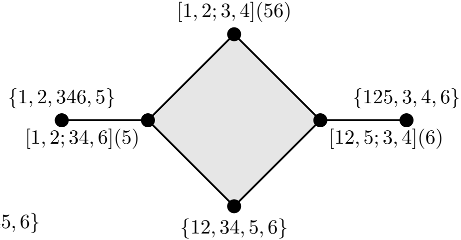

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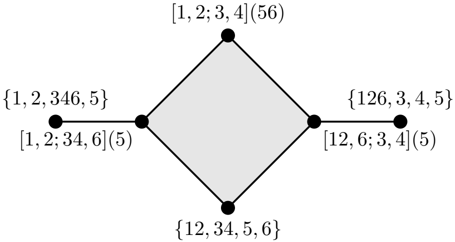

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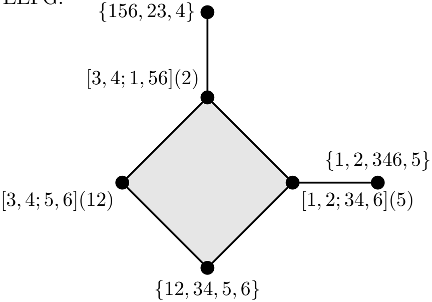

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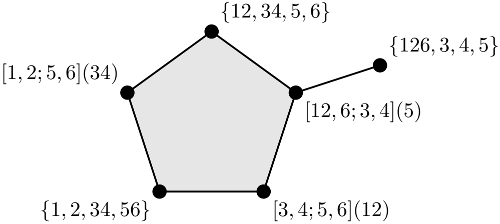

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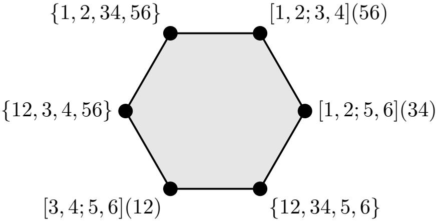

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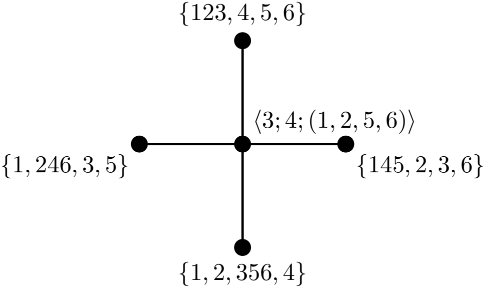

Tropicalized Planes versus Tropical Planes

.... is the same as ... Realizable Matroids versus All Matroids.

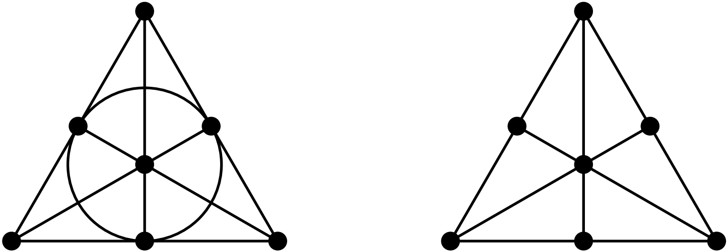

Figure: The point configurations for the Fano and non-Fano matroids.

Theorem

The Grassmannian $\mathrm{Gr}(3, n)$ *is a pure polyhedral complex of dimension* $2n - 9$*. The Dressian* $\mathrm{Dr}(3, n)$ *is not pure and it strictly contains* $\mathrm{Gr}(3, n)$ *for* $n \geq 7$*.*
The dimension of the Dressian $\mathrm{Dr}(3, n)$ *is of order* $\Theta(n^2)$*.*

Serious Computations

Theorem (GFan, cddlib, homology)

The tropical Grassmannian $\mathrm{Gr}(3,7)$ *is a simplicial complex with*

$$f\text{-vector} \;=\; (721, 16800, 124180, 386155, 522585, 252000)\,.$$

Its homology is free Abelian and concentrated in top dimension:

$$H_*\big(\mathrm{Gr}(3,7);\mathbb{Z}\big) \;=\; H_5\big(\mathrm{Gr}(3,7);\mathbb{Z}\big) \;=\; \mathbb{Z}^{7470}\,.$$

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Theorem (Polymake, homology)

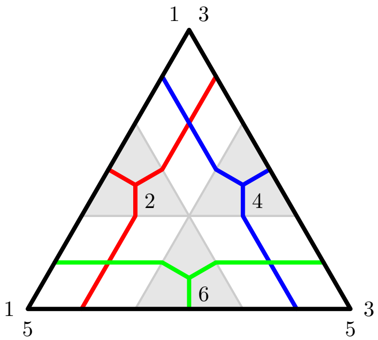
The Dressian $\mathrm{Dr}(3,7)$ *is a* 6*-dimensional polyhedral complex with*

$$f\text{-vector} \;=\; (616, 13860, 101185, 315070, 431025, 211365, 30)\,.$$

Its 5*-skeleton is triangulated by the Grassmannian* $\mathrm{Gr}(3,7)$*, and*

$$H_*(\mathrm{Dr}(3,7);\mathbb{Z}) \;=\; H_5(\mathrm{Dr}(3,7);\mathbb{Z}) \;=\; \mathbb{Z}^{7440}\,.$$

Serious Pictures



Mixed subdivisions of the triangle of side length $n - 3$ determine metric arrangements of n trees. In the picture we have $n = 6$.

Drawing The Unbounded Part of a Tropical Plane

Let $n \geq 4$ and consider an n-tuple of metric trees
$T = (T_1, T_2, \ldots, T_n)$ where T_i has the set of leaves $[n]\backslash\{i\}$.

A *metric tree* T_i comes with with non-negative edge lengths
By adding lengths along paths, the tree T_i defines a metric

$$\delta_i : ([n]\backslash\{i\}) \times ([n]\backslash\{i\}) \to \mathbb{R}_{\geq 0}.$$

An n-tuple T of metric trees is an *arrangement of metric trees* if

$$\delta_i(j,k) \ = \ \delta_j(k,i) \ = \ \delta_k(i,j) \qquad \text{for all } i,j,k \in [n].$$

Theorem

The combinatorial types of tropical planes in $\mathbb{TP}^{n-1}$ (i.e. cells in $\mathrm{Dr}(3,n)$) are in bijection with the arrangements of n metric trees.

Some Tree Arrangements are Not Metrizable

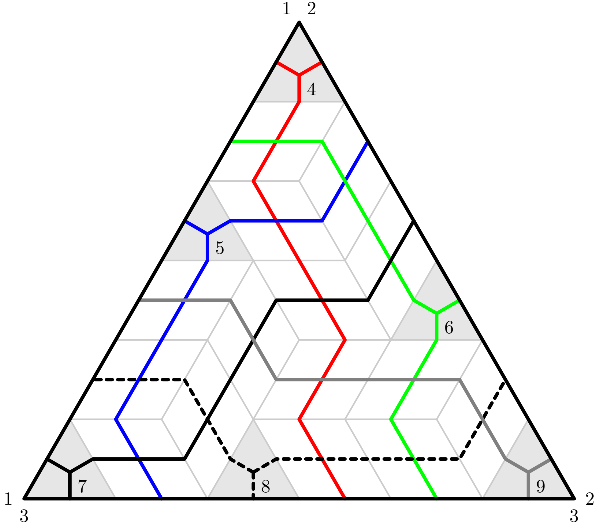


Figure: Abstract arrangement of nine caterpillar trees on eight leaves encoding a matroid subdivision that does not come from a tropical plane.