Orders of derivatives in differential Nullstellensatz

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The Problem of Consistency

Given a system of polynomial PDE, e.g.:

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 0 \end{cases}$$

Question: Is it consistent, i.e., does it have solutions?

(We look for solutions in differential extensions of the coefficient field . . .)

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Answer: YES

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Question: Is it consistent, i.e., does it have solutions?

(We look for solutions in differential extensions of the coefficient field . . .)

Answer: NO

Corresponding algebraic system

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 0 \end{cases} \longleftrightarrow \begin{cases} z_1 + z_2 = 0 \\ z_3 - z_4 = 0 \\ (z_5 + z_6)^2 + (z_7 + z_8)^2 = 0 \end{cases}$$

PDE system is consistent

Algebraic system is consistent

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PDE system is *in*consistent

Algebraic system is consistent

The converse is not always true.

Differential Nullstellensatz

Notation $F^{(\leq k)}$ set of all partial derivatives of elements of F of order $\leq k$

Theorem Polynomial PDE system F = 0 has no solutions

$$\label{eq:lambda} \begin{array}{c} & & \\ \\ \exists \ k \geq 0 \quad \text{ such that } \quad 1 \in \left\langle F^{(\leq k)} \right\rangle. \end{array}$$

Example

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1 \end{cases} \qquad k = 1$$

The Problem

Given non-negative integers m, n, h, d.

Find k(m, n, h, d) such that:

Polynomial PDE system F=0 in m independent variables, n dependent variables of order h and degree d has no solutions



$$1 \in \left\langle F^{\left(\leq k(m,n,h,d)\right)} \right\rangle$$

Main Result

[Seidenberg, 1956] Proposed to analyse the differential elimination algorithm to obtain the bound.

Theorem [GKOS '08]

$$k(m, n, h, d) = A(m + 8, \max(n, h, d)).$$

Here A(m, n) is the Ackermann function. It is not primitive recursive: grows faster than any such function.

Primitive recursive functions

Obtained from

- constant 0,
- successor S(k) = k + 1,
- projections $P_i^n(x_1,\ldots,x_n)=x_i$

by

- composition,
- primitive recursion: given primitive recursive $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_{n+2})$, define $h(x_1, \ldots, x_{n+1})$ by

$$h(0, x_1, \dots, x_n) = f(x_1, \dots, x_n),$$

 $h(m+1, x_1, \dots, x_n) = g(h(m, x_1, \dots, x_n), m, x_1, \dots, x_n).$

Ackermann Function

Definition

$$A(0,n) = n+1$$

 $A(m+1,0) = A(m,1)$
 $A(m+1,n+1) = A(m,A(m+1,n)).$

First few values [Wikipedia]:

	0	1	2	3	4	n
0	1	2	3	4	5	n+1
1	2	3	4	5	6	n+2
2	3	5	7	9	11	2n + 3
3	5	13	29	61	125	$2^{n+3}-3$
4	13	65533	$2^{65536} - 3$	$2^{2^{65536}} - 3$	A(3, A(4,3))	$\underbrace{2^{2}}^{\cdot \cdot 2}$ $n+3 \text{ twos}$
5	65533	$\underbrace{2^{2}}^{\cdot \cdot 2}$ 65535 twos	A(4,A(5,1))	A(4, A(5, 2))	A(4, A(5,3))	A(4,A(5,n-1))
m						

Why Ackermann Function?

Definition. A sequence of non-negative integer n-tuples τ_1, τ_2, \ldots is called **dicksonian**, if for all i < j, $\tau_j - \tau_i$ has at least one negative coordinate.

Alternative definition: a sequence of monomials u_1, u_2, \ldots such that $u_i \not\mid u_j$ for i < j.

Lemma [Dickson] Every dicksonian sequence terminates.

Lemma [G. Socias, 1991] Every dicksonian sequence of n-tuples, in which the maximal coordinate at each step increases by 1, has length at most

$$A(n, m-1) - 1,$$

where m is the maximal coordinate of the first tuple.

Why Dicksonian Sequences?

Polynomial *completion* algorithms such as

```
Algorithm Buchberger (F, \leq)
repeat
R := \mathsf{NormalForm}(\mathsf{SPoly}(F), F, \leq) \setminus \{0\}
F := F \cup R
until R = \varnothing
return F
```

produce dicksonian sequences of leading monomials.

Why Dicksonian Sequences?

Differential-algebraic completion algorithms, when applied to polynomial PDE systems, produce sequences of powers of $leading \ partial \ derivatives$ of the form

$$\left(\frac{\partial^h u_j}{\partial x_1^{i_1} \dots \partial x_m^{i_m}}\right)^d$$

such that the corresponding (m+n)-tuples

$$(i_1,\ldots,i_m,0,\ldots,d,\ldots,0)$$

form dicksonian sequences.

How Fast Do Tuples Grow?

Polynomial case: at each iteration, degree doubles (at most).

Differential case: at each iteration,

- order h doubles
- degree d becomes $(4d)^{\binom{2h+m}{m}+1}$ [GKOS '08].

We have a dicksonian sequence of (m+n)-tuples, in which the coordinates of the i-th tuple are bounded by a certain function f(i).

How Long Are these Sequences?

Function f(i) is not growing too fast: $\exists \delta$ such that $\forall i$

$$f(i+1) - f(i) \le A(\delta, f(i) - 1).$$

Lemma [GKOS'08] Length of such sequence does not exceed

$$[f^{-1}(A[m+n+\delta, f(1)-1])]$$

and the coordinates of the last tuple do not exceed

$$A(m + n + \delta, f(1) - 1).$$

Complexity of Differential Completion

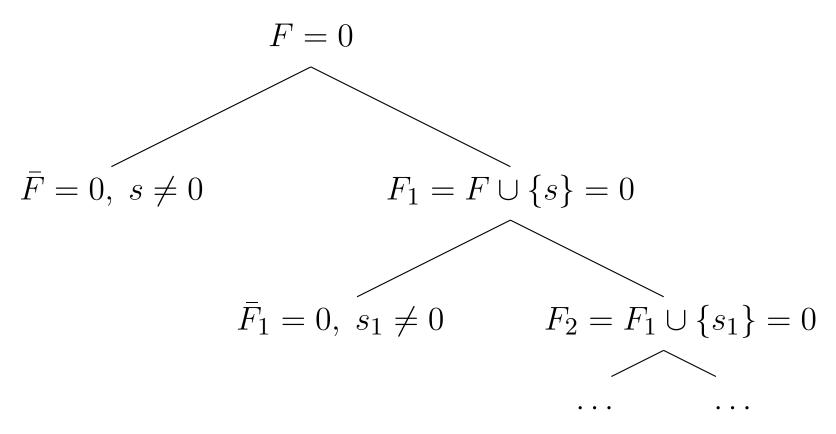
The number of iterations is bounded by:

$$A(m+n+3, \max(9, 2^h, d) - 1)$$

and the orders and degrees of the output, as well as the number of differentiations required to produce it, by:

$$k_{\text{completion}} = A(m + n + 4, \max(3, h, d) - 1).$$

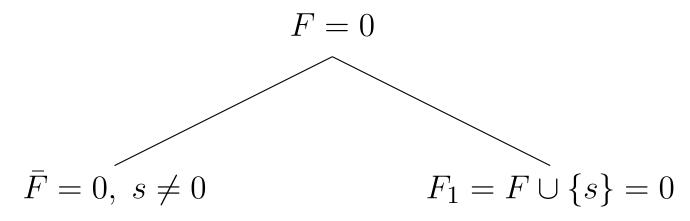
Splitting



Walking down the tree produces a dicksonian sequence

Obtaining Expression for 1

Let F = 0 be an inconsistent system.



$$\bar{F} \subset \left\langle F^{(\leq k_{\text{completion}})} \right\rangle$$

$$s \in \sqrt{\langle \bar{F} \rangle}$$

Suppose that we "recursively" obtained bound k_1 for F_1 . Then $1 \in \left\langle F^{(\leq k_1)} \right\rangle + \left\langle s^{(\leq k_1)} \right\rangle$. And sufficiently large powers of derivatives of s can be expressed

in terms of derivatives of \bar{F} .

Expressing Powers of Derivatives

- $s \in \sqrt{\langle \bar{F} \rangle}$
- [Kollar, 1988] A bound for q such that

$$s^q \in \langle \bar{F} \rangle$$

in terms of number of variables and degrees of \bar{F} .

• [Ritt] If $s^q \in \langle G \rangle$, then

$$\left(\frac{\partial s}{\partial x_i}\right)^{2q-1} \in \left\langle G^{(\leq q)} \right\rangle.$$

Proof for q = 2:

$$s^{2} \in \langle G \rangle \implies ss' \in \langle G, G' \rangle \implies (s')^{2} + ss'' \in \langle G, G', G'' \rangle \implies (s')^{3} \in \langle G, G', G'' \rangle.$$