Tensor decompositions and cubic sections of rational surface scrolls

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Seattle, January 6. 2016

Report on recent collaboration with

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arXiv:1601.00694

Let $X \subset \mathbb{P}^N$ be a smooth variety and let

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Definition

Let $y \in \mathbb{P}^N$. The 'Variety of aPolar Subschemes"

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When X is the d-uple embedding, then $\mathrm{VPS}_X(y, r)$ coincides with the variety $\mathrm{VSP}(f, r)$ of powersum decompositions of a homogeneous form f of degree d.

Definition

A subscheme $Z \subseteq X$ is said to be apolar to y, if

$$y \in \langle Z \rangle \subseteq \mathbb{P}^N$$

Note that $\mathrm{VPS}_X(y, r)$, by definition, contains only apolar subschemes that are in the closure of the set of smooth apolar subschemes.

Cox ring

To effectively study apolar subschemes we use the Cox ring of X:

$$Cox(X) = \oplus_{L \in \mathrm{Pic}(X)} H^0(X, L)$$

with multiplication

$$H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L').$$

Note that the Cow ring is graded by Pic(X).

Let

$$S \cong T \cong Cox(X)$$

For each element $A \in \mathrm{Pic}(X)$ we let

$$S_A = T_A^\vee :$$

A very ample $A \in \mathrm{Pic}(X)$ embeds $X \subset \mathbb{P}(S_A)$.

For

$$f \in S_A, \quad H_f := \{g | g(f) = 0\} \subset T_A$$

For each $B \in \mathrm{Pic}(X)$, we define

$$I_{f,B} = \begin{cases} (H_f : T_{A-B}) = \{g \in T_B : \ g \cdot T_{A-B} \subseteq H_f\}, & \text{if } A - B > 0 \\ T_B, & \text{otherwise,} \end{cases}$$

where $A - B > 0$ if the line bundle $A - B$ has global sections.

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We set

$$I_f := \bigoplus_{B \in \mathrm{Pic}(X)} I_{f,B} \subset T.$$

Similarly, a subscheme $Z \subset X$ has ideal

$$I_Z := \bigoplus_{B \in \mathrm{Pic}(X)} I_{Z,B} \subset T; \qquad I_{Z,B} = \{g \in T_B | g_{|Z} \equiv 0\}$$

apolarity lemma

Lemma

A subscheme $Z \subset X$ *is apolar to* $[f] \in \mathbb{P}(S_A)$*, if and only if* $I_Z \subset I_f$

Note: If X is a toric variety, $Cox(X)$ is a polynomial ring!

Rational curves

Sylvester (1850):

Proposition

Let $C \subset \mathbb{P}^N$ be a rational normal curve of degree N, and let $y \in \mathbb{P}^N$ be a general point. Then

$$\mathrm{VPS}_C(y, r) = 1 \text{ pt} \qquad \text{if } N = 2r - 1,$$

and

$$\mathrm{VPS}_C(y, r) = \mathbb{P}^1 \qquad \text{if } N = 2r - 2.$$

Elliptic curves

Following Room we show:

Proposition

Let $C \subset \mathbb{P}^N$ be an elliptic normal curve of degree $N+1$, and let $y \in \mathbb{P}^N$ be a general point. Then

$$\mathrm{VPS}_C(y,r) = 2 \text{ pts} \qquad \text{if } N = 2r-1$$

and

$$\mathrm{VPS}(y,r) = C \qquad \text{if } N = 2r-2$$

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What about curves of higher genus?

Toric surfaces

The following are examples with toric surfaces X.

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If $X = \mathbb{P}^2$, then $Pic(X) = \mathbb{Z}$ and $Cox(X) \cong \mathbb{C}[x_0, x_1, x_2]$

Hilbert, Mukai:

- $d = (A =)2 : \mathrm{VSP}(f, 3) = V_5$ (a Fano threefold)
- $d = (A =)3 : \mathrm{VSP}(f, 4) = \mathbb{P}^2$
- $d = (A =)4 : \mathrm{VSP}(f, 6) = V_{22}$ (a Fano threefold)
- $d = (A =)5 : \mathrm{VSP}(f, 7) = 1pt$
- $d = (A =)6 : \mathrm{VSP}(f, 10) = S$ (a K3 surface)

$\mathbb{P}^1 \times \mathbb{P}^1$

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathrm{Pic}(X) = \mathbb{Z} \times \mathbb{Z}$, and $Cox(X) \cong \mathbb{C}[x_0, x_1][y_0, y_1]$

If $A = (1, 1)$ and $f \in S_A$ is general, then $\mathrm{VPS}_X([f], 2) = \mathbb{P}^2$.

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Theorem

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $A = (2, 2) \in \mathrm{Pic}(X)$ and $f \in S_A$ be a general section. $\mathrm{VPS}_{\mathbb{P}^1 \times \mathbb{P}^1}([f], , 4)$ is isomorphic to a smooth quadric threefold blown up along a smooth rational normal curve.

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Idea of proof:

- $\dim I_{f,(2,1)} = 4$.
- If $[\Gamma] \in \mathrm{VPS}_{\mathbb{P}^1\times\mathbb{P}^1}([f], 4)$, then $\dim I_{\Gamma,(1,2)} = 2$, and $I_{\Gamma,(1,2)} \subset I_{f,(2,1)}$.
- Therefore there is a natural map,
  $\Phi_f : \mathrm{VPS}_{\mathbb{P}^1\times\mathbb{P}^1}([f], , 4) \to G(2, I_{f,(2,1)})$.
- If $g \in I_{f,(2,1)})$ is general, then $Z(g) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a rational curve apolar to f.
- Use Sylvesters VPS-result on rational curves to show that $\mathrm{Im}\Phi_f$ is a smooth quadric threefold.

Theorem

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $A = (3,3) \in \mathrm{Pic}(X)$ *and* $f \in S_A$ *be a general section. Then* $\mathrm{VPS}_{\mathbb{P}^1 \times \mathbb{P}^1}([f],,6)$ *is a surface isomorphic to a smooth Del Pezzo surface of degree* 5.

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Idea of proof:

- As a $(3,3)$ form f is the restriction to $Q := \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ of a form cubic form F.
- F has a unique apolar set of 5 points Γ_0, and $\Gamma_0 \cap Q = \emptyset$.
- If $\Gamma \subset Q$ is 6 general points, then $\Gamma = Q \cap C$ for a twisted cubic curve C.
- $\mathrm{VPS}_{\mathbb{P}^1 \times \mathbb{P}^1}([f], , 6) = \mathit{Hilb}_{3t+1}(\Gamma_0)$ the Hilbert scheme of twisted cubic curves that contains Γ_0.

F_1

Let $X = F_1$, then $\mathrm{Pic}(X) = \mathbb{Z} \times \mathbb{Z} = \langle E, F \rangle$, $E^2 = -1$, $E \cdot F = 1$, $F^2 = 0$.

If $A = E + 2F$ and $f \in S_A$ is general, then $\mathrm{VPS}_X([f], 2) = \mathbb{P}^1$

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Theorem

Let $X = F_1$, $A = 3E + 6F$ and $f \in S_A$ a general section. Then $\mathrm{VPS}_{F_1}([f], , 8)$ is isomorphic to $\mathbb{P}^2$ blown up in 8 points.

Idea of proof:

- $\dim I_{f,(2E+3F)} = 2$.
- If $g \in I_{f,(2E+3F)}$ is general, then $Z(g) \subset F_1$ is an elliptic curve C_g.
- Any $\Gamma \in \mathrm{VPS}_{F_1}(f, 8)$ is contained in C_g for some g.
- Use VPS-result on elliptic curves to show that

$$\cup_g \mathrm{VPS}_{C_g}(f, 8) = \cup_g C_g \to \mathrm{VPS}_{F_1}([f], , 8)$$

is a birational morphism.

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Thank You!