Secants of the Veronese and the Determinant

Cameron Farnsworth

Texas A&M University

JMM 2016 Seattle, WA

Basic notation and definitions

- V: Complex vector space
- $S^d V$: homogeneous degree d polynomials on V^*
- The **Veronese variety** is the image of the map

$$v_d : \mathbb{P}V \longrightarrow \mathbb{P}S^d V$$
$$[x] \mapsto [x^d].$$

- The variety

$$\sigma_r(v_d(\mathbb{P}V)) = \overline{\bigcup_{p_1,\ldots,p_r \in v_d(\mathbb{P}V)} \langle p_1, \ldots, p_r \rangle} \subseteq \mathbb{P}S^d V$$

is called the **r-th secant variety of the Veronese variety**.

- Let $P \in S^d V$. $\underline{R}_S(P) :=$ the minimal r such that $P \in \sigma_r(v_d(\mathbb{P}V))$. $\underline{R}_S(P)$ is called the **symmetric border rank of P**.

Basic notation and definitions

- V: Complex vector space
- S^dV: homogeneous degree d polynomials on V^*
- The **Veronese variety** is the image of the map

$$v_d : \mathbb{P}V \longrightarrow \mathbb{P}S^dV$$
$$[x] \mapsto [x^d].$$

- The variety

$$\sigma_r(v_d(\mathbb{P}V)) = \overline{\bigcup_{p_1,\ldots,p_r \in v_d(\mathbb{P}V)} \langle p_1, \ldots, p_r \rangle} \subseteq \mathbb{P}S^dV$$

is called the r**-th secant variety of the Veronese variety**.

- Let $P \in S^dV$. $\underline{R}_S(P) :=$ the minimal r such that $P \in \sigma_r(v_d(\mathbb{P}V))$. $\underline{R}_S(P)$ is called the **symmetric border rank of** P.

Basic notation and definitions

- V: Complex vector space
- $S^d V$: homogeneous degree d polynomials on V^*
- The **Veronese variety** is the image of the map

$$v_d : \mathbb{P}V \longrightarrow \mathbb{P}S^d V$$
$$[x] \mapsto [x^d].$$

- The variety

$$\sigma_r(v_d(\mathbb{P}V)) = \overline{\bigcup_{p_1,\dots,p_r \in v_d(\mathbb{P}V)} \langle p_1, \dots, p_r \rangle} \subseteq \mathbb{P}S^d V$$

is called the **r-th secant variety of the Veronese variety**.

- Let $P \in S^d V$. $\underline{R}_S(P) :=$ the minimal r such that $P \in \sigma_r(v_d(\mathbb{P}V))$. $\underline{R}_S(P)$ is called the **symmetric border rank of** P.

Basic notation and definitions

- V: Complex vector space
- $S^d V$: homogeneous degree d polynomials on V^*

- The **Veronese variety** is the image of the map

$$v_d : \mathbb{P}V \longrightarrow \mathbb{P}S^d V$$
$$[x] \mapsto [x^d].$$

- The variety

$$\sigma_r(v_d(\mathbb{P}V)) = \overline{\bigcup_{p_1,\ldots,p_r \in v_d(\mathbb{P}V)} \langle p_1, \ldots, p_r \rangle} \subseteq \mathbb{P}S^d V$$

is called the **r-th secant variety of the Veronese variety**.

- Let $P \in S^d V$. $\underline{R}_S(P) :=$ the minimal r such that $P \in \sigma_r(v_d(\mathbb{P}V))$. $\underline{R}_S(P)$ is called the **symmetric border rank of P**.

The polynomial Waring problem

- **Goal:** Exhibit new lower bounds on symmetric border rank of the determinant polynomial.
- **Motivation**
  - **The polynomial Waring problem** asks given $P \in S^d V$, how many powers of linear forms must be added to equal P?
  - Nonmembership of P in the r-th secant variety of the Veronese variety demonstrates

$$P \neq \ell_1^d + \ldots + \ell_r^d$$

    where $\ell_i \in V$.

The polynomial Waring problem

- **Goal:** Exhibit new lower bounds on symmetric border rank of the determinant polynomial.
- **Motivation**
  - **The polynomial Waring problem** asks given $P \in S^d V$, how many powers of linear forms must be added to equal P?
  - Nonmembership of P in the r-th secant variety of the Veronese variety demonstrates

$$P \neq \ell_1^d + \ldots + \ell_r^d$$

where $\ell_i \in V$.

The polynomial Waring problem

- **Goal:** Exhibit new lower bounds on symmetric border rank of the determinant polynomial.
- **Motivation**
  - **The polynomial Waring problem** asks given $P \in S^d V$, how many powers of linear forms must be added to equal P?
  - Nonmembership of P in the r-th secant variety of the Veronese variety demonstrates

$$P \neq \ell_1^d + \ldots + \ell_r^d$$

where $\ell_i \in V$.

Standard flattenings

- Regard $S^k V^*$ as the span of
$$\frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$
- For $\alpha \in S^k V^*$ and $P \in S^d V$, $\alpha \bullet P$ denotes differentiation.
- For $P \in S^d V$, Sylvester defined linear maps
$$P_{k,d-k} : S^k V^* \longrightarrow S^{d-k} V$$
$$\alpha \mapsto \alpha \bullet P.$$
- We will call such a map a **(k-th) standard flattening of P.**

Standard flattenings

- Regard $S^k V^*$ as the span of

$$\frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

- For $\alpha \in S^k V^*$ and $P \in S^d V$, $\alpha \bullet P$ denotes differentiation.
- For $P \in S^d V$, Sylvester defined linear maps

$$P_{k,d-k} : S^k V^* \longrightarrow S^{d-k} V$$
$$\alpha \mapsto \alpha \bullet P.$$

- We will call such a map a **(k-th) standard flattening of P.**

How do standard flattenings prove lower bounds?

- Our goal is to write P as
$$P = \ell_1^d + \ldots + \ell_r^d.$$
- Assuming we have such decomposition, then we have
$$P_{k,d-k} = [\ell_1^d]_{k,d-k} + \ldots + [\ell_r^d]_{k,d-k}$$
- Thus
$$\text{rank}(P_{k,d-k}) \leq \text{rank}([\ell_1^d]_{k,d-k}) + \cdots + \text{rank}([\ell_r^d]_{k,d-k})$$
- If $\text{rank}([\ell_i^d]_{k,d-k}) = t$, then
$$\underline{R}_S(P) \geq \frac{\text{rank}(P_{k,d-k})}{t}.$$

How do standard flattenings prove lower bounds?

- Our goal is to write P as
$$P = \ell_1^d + \ldots + \ell_r^d.$$
- Assuming we have such decomposition, then we have
$$P_{k,d-k} = [\ell_1^d]_{k,d-k} + \ldots + [\ell_r^d]_{k,d-k}$$
- Thus
$$\operatorname{rank}(P_{k,d-k}) \leq \operatorname{rank}([\ell_1^d]_{k,d-k}) + \cdots + \operatorname{rank}([\ell_r^d]_{k,d-k})$$
- If $\operatorname{rank}([\ell_i^d]_{k,d-k}) = t$, then
$$\underline{R}_S(P) \geq \frac{\operatorname{rank}(P_{k,d-k})}{t}.$$

Lower bounds for the determinant from standard flattenings

Let $\det_n$ denote the polynomial obtained by taking the determinant of an $n \times n$ matrix of indeterminates.

$$\underline{R}_S(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2.$$

Or,

$$\underline{R}_S(\det_n) \gtrsim \frac{2 \cdot 4^n}{\pi \cdot n}.$$

Lower bounds for the determinant from standard flattenings

Let $\det_n$ denote the polynomial obtained by taking the determinant of an $n \times n$ matrix of indeterminates.

$$\underline{R}_S(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2.$$

Or,

$$\underline{R}_S(\det_n) \gtrsim \frac{2 \cdot 4^n}{\pi \cdot n}.$$

New Results

Theorem[F.]

Let $n \geq 5$.
For n even:

$$\underline{R}_S(\det_n) \geq \left(1 + \frac{8(-8+6n^2+n^3)}{(-1+n)(2+n)(4+n)^2(-2+n^2)}\right)\binom{n}{\frac{n}{2}}^2.$$

For n odd:

$$\underline{R}_S(\det_n) \geq \left(1 + \frac{16(9+8n+n^2)}{(3+n)(5+n)^2(-2+n^2)}\right)\binom{n}{\frac{n-1}{2}}^2.$$

Or,

$$\underline{R}_S(\det_n) \gtrsim \frac{2^{2n+1}}{\pi \cdot n} + \frac{2^{2n+1}}{\pi \cdot n^4}.$$

For example, the old lower bound gives $\underline{R}_S(\det_5) \geq 100$; whereas, the lower bound stated above shows $\underline{R}_S(\det_5) \geq 107$.

New Results

Theorem[F.]

Let $n \geq 5$.
For n even:

$$\underline{R}_S(\det_n) \geq \left(1 + \frac{8(-8+6n^2+n^3)}{(-1+n)(2+n)(4+n)^2(-2+n^2)}\right)\binom{n}{\frac{n}{2}}^2.$$

For n odd:

$$\underline{R}_S(\det_n) \geq \left(1 + \frac{16(9+8n+n^2)}{(3+n)(5+n)^2(-2+n^2)}\right)\binom{n}{\frac{n-1}{2}}^2.$$

Or,

$$\underline{R}_S(\det_n) \gtrsim \frac{2^{2n+1}}{\pi \cdot n} + \frac{2^{2n+1}}{\pi \cdot n^4}.$$

For example, the old lower bound gives $\underline{R}_S(\det_5) \geq 100$; whereas, the lower bound stated above shows $\underline{R}_S(\det_5) \geq 107$.

More results

Theorem[F.]

$$\underline{R}_S(\det_3) \geq 14 \text{ and } \underline{R}_S(\mathrm{perm}_3) \geq 14.$$

Corollary

$$14 \leq \underline{R}_S(\mathrm{perm}_3) \leq 16$$

More results

Theorem[F.]

$$\underline{R}_S(\det_3) \geq 14 \text{ and } \underline{R}_S(\mathrm{perm}_3) \geq 14.$$

Corollary

$$14 \leq \underline{R}_S(\mathrm{perm}_3) \leq 16$$

Sketch of proof

In [LO13], Landsberg and Ottaviani defined linear maps called **Young Flattenings** associated to a homogeneous polynomial $P \in S^d V$

$$\mathcal{F}_{\lambda,\mu}(P) : S_\lambda V \longrightarrow S_\mu V$$

where $S_\lambda V$ and $S_\mu V$ are irreducible $GL(V)$-modules such that the partitions λ and μ satisfy certain conditions.

Proposition 4.1 of [LO13]

Let $P \in S^d(V)$ and $[x] \in \mathbb{P}V$, then

$$\underline{R}_S(P) \geq \frac{\text{rank}(\mathcal{F}_{\lambda,\mu}(P))}{\text{rank}(\mathcal{F}_{\lambda,\mu}(x^d))}.$$

Sketch of proof

In [LO13], Landsberg and Ottaviani defined linear maps called **Young Flattenings** associated to a homogeneous polynomial $P \in S^d V$

$$\mathcal{F}_{\lambda,\mu}(P) : S_\lambda V \longrightarrow S_\mu V$$

where $S_\lambda V$ and $S_\mu V$ are irreducible $GL(V)$-modules such that the partitions λ and μ satisfy certain conditions.

Proposition 4.1 of [LO13]

Let $P \in S^d(V)$ and $[x] \in \mathbb{P}V$, then

$$\underline{R}_S(P) \geq \frac{\text{rank}(\mathcal{F}_{\lambda,\mu}(P))}{\text{rank}(\mathcal{F}_{\lambda,\mu}(x^d))}.$$

Sketch of proof continued.

Let $V = \mathbb{C}^n \otimes \mathbb{C}^n$. Define a **Koszul-Young flattening** of the determinant to be the linear maps

$$(\det_n)^{\wedge 2}_{d,n-d} : S^d V^* \otimes \bigwedge^2 V \xrightarrow{(\det_n)_{d,n-d} \otimes Id_{\wedge^2 V}} S^{n-d} V \otimes \bigwedge^2 V \xrightarrow{d_2} S^{n-d-1} V \otimes \bigwedge^3 V$$

where d_2 is the map from the Koszul complex:

$$\cdots \xrightarrow{d_{k-1}} S^q V \otimes \bigwedge^k (V) \xrightarrow{d_k} S^{q-1} V \otimes \bigwedge^{k+1} (V) \xrightarrow{d_{k+1}} \cdots .$$

References I

- Cameron Farnsworth, **Koszul-Young flattenings and symmetric border rank of the determinant**, Journal of Algebra **447** (2016), 664 – 676.
- J.M. Landsberg and Giorgio Ottaviani, **Equations for secant varieties of veronese and other varieties**, Annali di Matematica Pura ed Applicata **192** (2013), no. 4, 569–606 (English).

Thank you for your attention.