Normality of secant varieties

Brooke Ullery

Joint Mathematics Meetings

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Introduction

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Definition

The **secant variety** of X, $\Sigma(X, \mathcal{L}) = \Sigma \subset \mathbb{P}^r$ is the Zariski closure of the union of lines spanned by two points of X (i.e. secant and tangent lines).

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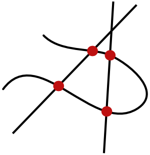

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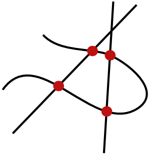

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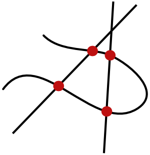

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Intuition/Motivating question

As $\mathcal{L}$ becomes more positive, the singularities of Σ should improve. How positive does $\mathcal{L}$ need to be for Σ to be normal?

Main results

Theorem

Let X be a smooth projective variety, $\mathcal{L}$ a 3-very ample line bundle on X, and m_x the ideal sheaf of $x \in X$. If for all $x \in X$ and $i > 0$, the map

$$Sym^i H^0(\mathcal{L} \otimes m_x^2) \to H^0(\mathcal{L}^{\otimes i} \otimes m_x^{2i})$$

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*Note that the surjectivity of these maps is the same as $\mathrm{bl}_x \mathcal{L}(-2E)$ being normally generated for all blowups at a point, where E is the exceptional divisor.

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Corollary 3: arbitrary dimension

If X is a variety of dimension n, $\mathcal{A}$ is very ample, $\mathcal{B}$ is nef, and

$$\mathcal{L} = K_X + 2(n+1)\mathcal{A} + \mathcal{B},$$

then $\Sigma(X, \mathcal{L})$ is normal.

Recent progress

Theorem (Chou, Song 2015)

With hypotheses from Corollary 3,

1. $\Sigma(X, \mathcal{L})$ *has Du Bois singularities, and*
2. $\Sigma(X, \mathcal{L})$ *has rational singularities* $\iff H^i(\mathcal{O}_X) = 0$ *for* $i > 0$.

Resolution of singularities construction (Schwarzenberger, Bertram, Vermeire)

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Recall: $X^{[2]}$ is smooth, and its universal subscheme is the incidence variety

$$\Phi = \{(x, Z) \in X \times X^{[2]} : x \in Z\} \cong bl_\Delta(X \times X).$$

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Define the two projections

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  which sends

  $$\left(Z, H^0(\mathcal{L}|_Z) \twoheadrightarrow Q\right) \mapsto \left(H^0(\mathcal{L}) \twoheadrightarrow Q\right),$$

  where Q is some one-dimensional quotient.

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In fact. . .

Lemma (Bertram (curves), Vermeire (higher dim))

Let $t : \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \to \Sigma$ be f with its target restricted. Then t is an isomorphism away from $t^{-1}(X)$. In particular, t is a resolution of singularities.

Geometry of the resolution

- **Pre-image of X?**

$$t^{-1}(X) = \{\big(Z, H^0(\mathcal{L}|_Z) \to H^0(\mathcal{L}|_x)\big|\, x \in Z\} \cong \Phi.$$

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To summarize:

$$\begin{array}{ccccccc}
\mathrm{bl}_x X & \hookrightarrow & \Phi & \hookrightarrow & \mathbb{P}(\mathcal{E}_{\mathcal{L}}) & & \\
\downarrow & & \downarrow q & & \downarrow t & \searrow f & \\
\{x\} & \hookrightarrow & X & \hookrightarrow & \Sigma(X, \mathcal{L}) & \hookrightarrow & \mathbb{P}^r
\end{array}$$

Strategy for proof of main theorem: Show that $t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_\mathcal{L})} = \mathcal{O}_\Sigma$ by checking on the completion at $x \in X$ (i.e. at the singular points).

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Conjecture

If X is a smooth projective curve of genus g, and $\mathcal{L}$ a line bundle such that

$$\deg \mathcal{L} \geq 2g + 2k + 1,$$

then $\Sigma_k(X, \mathcal{L})$ is normal.